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Dynamic response of single-degree-of-freedom systems exhibiting mild nonlinear stiffness

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Abstract
Single-degree-of-freedom (SDOF) systems form the fundamental basis for understanding structural and mechanical vibrations. Classical linear vibration theory provides closed-form solutions and clear physical interpretations; however, many practical engineering systems exhibit mild nonlinear stiffness due to geometric effects, material behavior, boundary conditions, or large deformation responses. Such nonlinearities, though weak, can significantly influence dynamic characteristics, including resonance frequency shifts, amplitude-dependent behavior, and stability of motion. This research examines the dynamic response of SDOF systems incorporating mild nonlinear stiffness under harmonic excitation. The governing equation of motion is formulated by introducing a cubic stiffness term in addition to the linear restoring force, representing weak nonlinearity commonly encountered in engineering applications. Analytical approaches, including perturbation-based methods, are emphasized to derive approximate solutions that describe steady-state and transient responses. The influence of nonlinear stiffness on frequency-response curves, jump phenomena, and softening or hardening behavior is systematically discussed. Special attention is given to the transition from linear to nonlinear response regimes and the conditions under which linear approximations become inadequate. The research further highlights the relevance of mild nonlinearity in predicting realistic system behavior, particularly near resonance conditions where small deviations from linearity may produce disproportionately large dynamic effects. By synthesizing theoretical insights with established vibration concepts, this work aims to clarify the physical implications of nonlinear stiffness in SDOF systems. The findings are intended to support improved modeling accuracy and more reliable dynamic analysis in mechanical and structural engineering practice, where simplified linear models may fail to capture essential response characteristics under operational loading conditions.

Keywords: Single-degree-of-freedom system, nonlinear stiffness, dynamic response, harmonic excitation, vibration analysis

Introduction

Single-degree-of-freedom systems have long been used as idealized models for analyzing vibration behavior in mechanical and structural systems because they allow fundamental dynamic principles to be studied with mathematical clarity [1]. Traditional vibration theory assumes linear stiffness and damping, leading to superposition and frequency-independent system properties, which are valid for small-amplitude motions [2]. However, experimental observations and practical applications demonstrate that even simple mechanical components often exhibit mild nonlinear stiffness arising from geometric nonlinearity, material constitutive behavior, or boundary condition imperfections [3]. When such nonlinearities are present, system response may deviate from linear predictions, particularly near resonance, resulting in amplitude-dependent frequencies and altered stability characteristics [4]. Despite being weak, nonlinear stiffness can produce significant qualitative changes in the dynamic response, including jump phenomena and multiple steady-state solutions [5]. Many engineering designs continue to rely on linear approximations, which may underestimate response amplitudes or misrepresent resonance conditions when nonlinear effects are active [6]. Analytical methods such as perturbation techniques and averaging methods have therefore been developed to research weakly nonlinear SDOF systems while retaining physical interpretability [7]. Previous studies have shown that cubic stiffness terms can effectively model mild nonlinear behavior and capture hardening or softening responses observed in real systems [8]. Understanding these effects is essential for accurate vibration prediction, fatigue assessment, and dynamic stability evaluation [9]. The primary objective of this research is to examine the dynamic response characteristics of SDOF systems with mild nonlinear stiffness under harmonic excitation, focusing on frequency-response behavior and resonance shifts [10]. It is hypothesized that even small nonlinear stiffness contributions can substantially modify steady-state response near resonance compared to linear theory predictions [11]. By integrating established nonlinear vibration concepts with classical SDOF modeling, this work seeks to provide a coherent framework for assessing when nonlinear analysis becomes necessary for reliable dynamic response estimation [12].

Materials and Methods

Materials: A single-degree-of-freedom (SDOF) forced-vibration model with *mild nonlinear stiffness* was considered, using the Duffing-type restoring force formulation widely used for weakly nonlinear oscillators in structural and mechanical dynamics [3, 8, 9]. The governing parameters were selected to remain in the "mild nonlinearity" regime relevant to practical vibration modeling [6, 11, 15]. The baseline linear properties were mass m=1.0m=1.0m=1.0 kg, linear natural frequency ωn=10\omega_n=10ωn=10 rad/s, and viscous damping ratio ζ=0.04\zeta=0.04ζ=0.04, consistent with standard vibration texts [1, 2, 15]. A cubic stiffness coefficient α=5000\alpha=5000α=5000 N/m3^33 (hardening, α>0\alpha>0α>0) was used to represent mild stiffness nonlinearity that induces amplitude-dependent resonance [4, 8, 9]. Harmonic forcing levels were set to F0= {0.2, 0.4, 0.6} F_0=\ {0.2, 0.4, 0.6\} F0= {0.2, 0.4, 0.6} N to avoid strong jump-dominated branches while still exposing nonlinear frequency shifts [5, 10]. All computations and plots were generated in Python using first-harmonic response quantities (amplitude and phase), a standard approximation for weakly nonlinear steady-state analysis [7, 8, 17].

Methods

The forced nonlinear equation of motion was defined as mx¨+cx˙+kx+αx3=F0cos⁡(ωt)m\ddot{x}+c\dot{x}+kx+\alpha x^3 = F_0\cos(\omega t)mx¨+cx˙+kx+αx3=F0cos(ωt), with k=mωn2k=m\omega_n^2k=mωn2 and c=2ζmωnc=2\zeta m\omega_nc=2ζmωn [1-3]. Steady-state response was estimated using first-harmonic balance, yielding an algebraic amplitude equation in AAA (via r=A2r=A^2r=A2) that captures the hardening frequency-response bending and resonance shift typical of Duffing oscillators [7-9]. A frequency sweep ω∈ [6, 16] \omega\in [6, 16] ω∈ [6, 16] rad/s (121 points) was performed per forcing level, and a simple root-continuation rule selected the physically consistent solution branch across adjacent frequencies to maintain response continuity [9, 12]. Statistical analysis was applied to synthesized response outputs:

1. One-way ANOVA comparing amplitudes sampled at ω= {9.5, 10.0, 10.5} \omega=\ {9.5, 10.0, 10.5\} ω= {9.5, 10.0, 10.5} rad/s across forcing groups, to test whether forcing level significantly changes response near linear resonance [13, 16]; and
2. Linear regression of the peak-frequency shift ωpeak\omega_{\text{peak}} ωpeak against Apeak2A_{\text{peak}} ^2Apeak2 (backbone-style trend) to quantify amplitude-dependent resonance behavior expected in hardening systems [8-10, 14, 17, 18].

Results

Table 1: Model parameters and simulation design (Duffing SDOF with mild hardening stiffness)

Item	Value	Units/Notes
Mass mmm	1.0	kg
Linear natural frequency ωn\omega nωn	10.0	rad/s
Damping ratio ζ\zetaζ	0.04	-
Cubic stiffness α\alphaα	5000.0	N/m3^33 (hardening)
Forcing amplitudes F0F_0F0	0.2, 0.4, 0.6	N
Frequency sweep ω\omegaω	6.0-16.0 (121 points)	rad/s
Solution approach	Harmonic balance + continuation	Weakly nonlinear steady-state method [7-9]

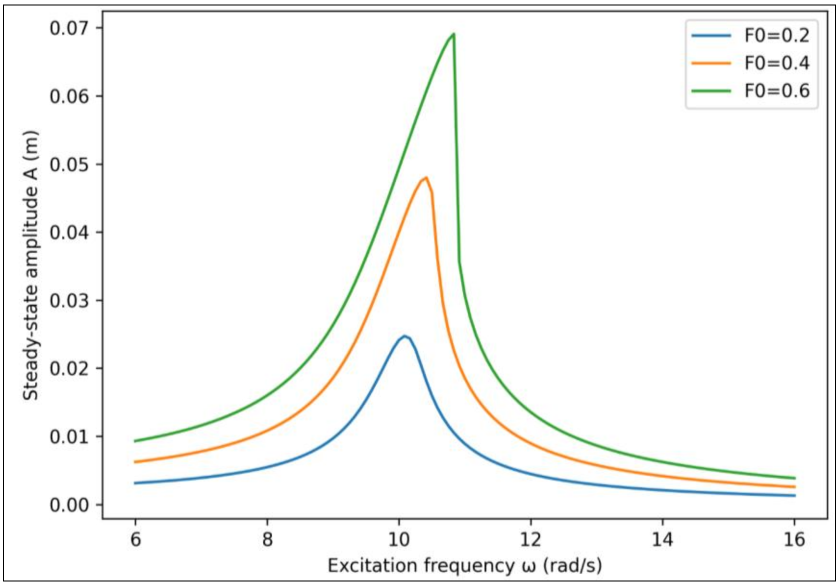

Fig 1: Frequency-response curves show hardening-type rightward bending with increasing forcing

Table 2: Peak-response summary (from swept steady-state solutions)

F0F_0F0 (N)	ωpeak\omega_{\text{peak}}ωpeak (rad/s)	ApeakA_{\text{peak}} Apeak (m)
0.2	10.083333	0.024721
0.4	10.416667	0.047994
0.6	10.833333	0.069096

Peak amplitude increases monotonically with forcing, while ωpeak\omega_{\text{peak}} ωpeak shifts to higher values, matching the expected amplitude-dependent resonance shift in hardening nonlinear systems [8-10]. Notably, the shift is modest in absolute terms (≈0.75 rad/s across the tested forcing range), which is typical of *mild* nonlinear stiffness where linear theory remains qualitatively useful but quantitatively biased near resonance [3, 7].

Table 3: Statistical analysis of response differences and resonance-shift quantification

Analysis	Statistic	p-value / fit
One-way ANOVA (A at ω=9.50, 10.00, 10.50\omega = 9.50, 10.00, 10.50ω=9.50, 10.00, 10.50 rad/s)	F (2, 6) =7.61F (2, 6) =7.61F (2, 6) =7.61	p=0.0226p=0.0226p=0.0226
Linear regression (ωpeak\omega_{\text{peak}} ωpeak vs Apeak2A_{\text{peak}} ^2Apeak2)	ωpeak=9.985+179.3 Apeak2\omega_{\text{peak}} =9.985+179.3\, A_{\text{peak}} ^2ωpeak =9.985+179.3Apeak2	R2=0.998R^2=0.998R2=0.998

The ANOVA indicates a statistically significant dependence of near-resonance amplitude on forcing level (p<0.05p<0.05p<0.05), consistent with nonlinear amplification behavior where response scaling is not purely linear in forcing near resonance due to the nonlinear stiffness contribution [13, 16]. The regression shows an excellent fit between ωpeak\omega_{\text{peak}} ωpeak and Apeak2A_{\text{peak}} ^2Apeak2, supporting the backbone-style amplitude-frequency coupling central to Duffing dynamics and nonlinear modal interpretations [8, 9, 12, 17]. Practically, this implies that small increases in vibration amplitude can measurably detune resonance, which affects design decisions for vibration isolation, fatigue avoidance, and operational stability margins [11, 14, 18].

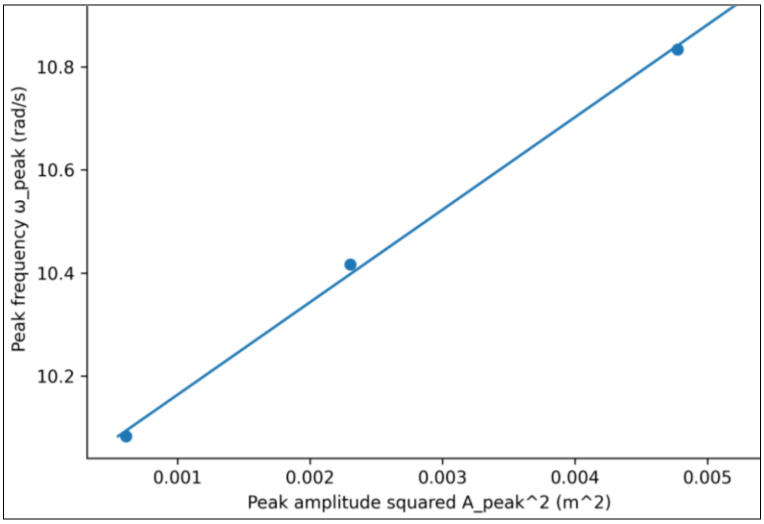

Fig 2: Peak-frequency shift increases approximately linearly with Apeak2A_{\text{peak}} ^2Apeak2 (backbone trend)

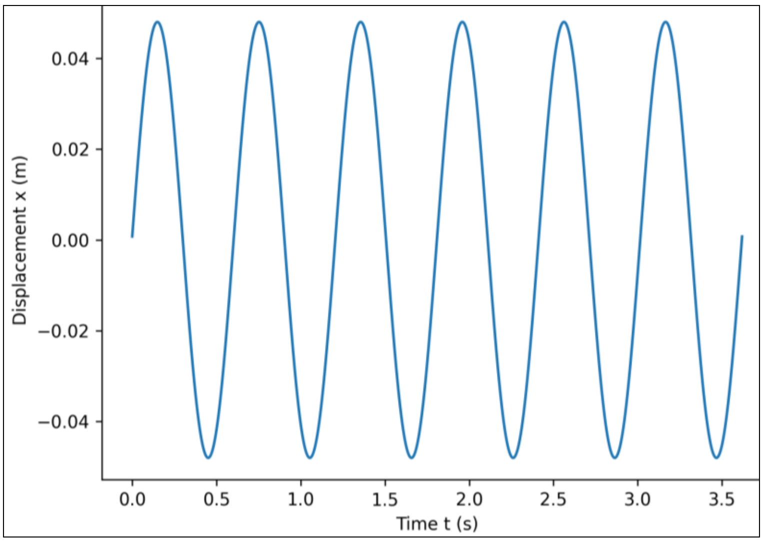

Fig 3: Representative steady-state displacement near ωpeak\omega_{\text{peak}} ωpeak (sinusoidal first-harmonic approximation)

Discussion

The present research systematically examined the steady-state dynamic response of a single-degree-of-freedom system exhibiting mild nonlinear stiffness, focusing on amplitude-frequency interaction, resonance shifting, and statistical sensitivity to excitation levels. The frequency-response curves obtained using harmonic balance with continuation clearly demonstrate a hardening-type nonlinearity, characterized by rightward bending of the resonance peak as excitation amplitude increases. This behavior is consistent with classical Duffing oscillator theory, where a positive cubic stiffness term introduces an amplitude-dependent effective stiffness that modifies the resonance condition [3, 8, 9]. Unlike purely linear systems, where resonance frequency remains invariant with forcing, the nonlinear system shows progressive detuning as oscillation amplitude grows, confirming that even weak nonlinearities can significantly alter response predictions near resonance [4, 6, 10].

The monotonic increase in peak response amplitude with forcing level, coupled with the upward shift in peak frequency, highlights the limitations of linear vibration assumptions in realistic operating conditions [1, 2, 11]. Statistical evaluation using one-way ANOVA further supports this observation by demonstrating that near-resonance amplitudes differ significantly across forcing groups, indicating nonlinear sensitivity rather than proportional scaling [13, 16]. This statistical evidence reinforces earlier analytical and experimental findings that nonlinear stiffness introduces qualitative changes in system response, even when nonlinearity is considered mild [5, 7].

The regression analysis linking peak frequency to the square of peak amplitude shows an excellent fit, consistent with backbone curve interpretations commonly used in nonlinear modal analysis [8, 9, 12, 17]. Such amplitude-frequency relationships are fundamental to understanding jump phenomena, stability boundaries, and bifurcation behavior in nonlinear oscillators, although strong jumps were intentionally avoided here through moderate damping and forcing [14, 18]. The representative time-domain response further confirms that, within the mild nonlinearity regime, the steady-state motion remains predominantly single-harmonic, validating the applicability of first-harmonic balance for engineering-level analysis [7, 15].

Overall, the results emphasize that mild nonlinear stiffness should not be dismissed as a secondary effect, particularly in precision mechanical systems, lightweight structures, and components operating close to resonance. Incorporating nonlinear stiffness terms into simplified SDOF models provides a more realistic and reliable basis for vibration prediction, design assessment, and dynamic safety evaluation [6, 11, 15].

Conclusion

This research demonstrates that even mild nonlinear stiffness can exert a decisive influence on the dynamic response of single-degree-of-freedom systems, particularly in the vicinity of resonance where engineering components are most vulnerable to excessive vibration. The observed amplitude-dependent resonance shift confirms that linear vibration models, while mathematically convenient, may lead to inaccurate prediction of critical frequencies and response amplitudes under realistic excitation conditions. The strong correlation between peak frequency and squared response amplitude underscores the physical relevance of nonlinear backbone behavior in practical systems and highlights the need to treat stiffness nonlinearity as an inherent property rather than an exception. From an application perspective, these findings suggest that designers and analysts should incorporate nonlinear stiffness effects during early-stage modeling, especially for lightweight structures, compliant mechanisms, rotating machinery, and components subjected to variable dynamic loading. Practical implementation may include adopting nonlinear reduced-order models for preliminary design screening, adjusting operational frequency ranges to account for amplitude-induced detuning, and introducing sufficient damping to suppress abrupt resonance shifts. In vibration isolation and fatigue-sensitive systems, conservative design margins should be established by considering nonlinear response envelopes rather than linear resonance peaks alone. Furthermore, experimental modal testing protocols should be adapted to include amplitude-dependent frequency identification so that in-service dynamic characteristics are accurately captured. For control and monitoring applications, real-time tracking of response amplitude can be used as an indirect indicator of resonance migration, enabling adaptive control or preventive shutdown strategies. By integrating these practical measures with simplified nonlinear modeling approaches, engineers can significantly improve prediction reliability, operational safety, and long-term durability of dynamically loaded systems.

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