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An algorithm based on power series and shifted chebyshev polynomials for solving volterra and fredholm integral equations

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Abstract
This paper considered an approximate solution of linear second-order integral equations of Volterra and Fredholm types using power series and shifted Chebyshev polynomials as basis functions. In addition, shifted Chebyshev Gauss-Lobatto collocation points were chosen to collocate the approximate solution and numerical examples were performed on some problems using both power series and shifted Chebyshev polynomials for various orders in terms of errors obtained. The two basis functions were compared. However, as the table of errors shows shifted Chebyshev polynomials outperformed power series in some problems.

Keywords: Chebyshev gauss-lobatto, chebyshev polynomial, collocation, approximate solution and volterra and fredholm, integral equation

Introduction

Integral equations serve as a fundamental tool in mathematical physics, often arising as reformulations of problems involving partial differential equations or ordinary differential equations. In the absence of analytical solutions or when obtaining them proves excessively challenging, numerical simulation has emerged as a powerful approach for modeling physical phenomena. Therefore, the study of integral equations and the development of efficient numerical methods to solve them are crucial in practical applications. Numerous researchers have dedicated their efforts to the numerical solution of Volterra and Fredholm integral equations, employing diverse techniques including the least squares method, Chebyshev polynomials, scaling function and interpolation method, separation of variables method, Hosoya polynomial, Legendre polynomial, hybrid methods, and other innovative approaches. This manuscript aims to provide a comprehensive review of the existing literature, highlighting the significance of numerical methods for solving integral equations in the context of mathematical physics and engineering science.

In a recent study conducted by Ayinde, James, Ishaq, and Oyedepo (2022) [4], the researchers explored the numerical advantages offered by various types of polynomials, including Taylor, Chebyshev, Hermite, Legendre, and Laguerre polynomials, in the context of solving integral and integro-differential equations (IDEs). Notably, the team successfully utilized third kind Chebyshev polynomials to solve high-order integro-differential equations, showcasing their efficacy in the field. Shoukrallah and Elghory (2021) [13] also introduced an innovative approach for solving second-kind Fredholm integral equations, employing shifted Legendre polynomials. Their method revolves around utilizing these polynomials in a matrix form to efficiently address the integral equations.

On the other hand, Ratinan, Ampol, and Phansphitcha (2020) [12] conducted a study focusing on the direct and inverse problems associated with one-dimensional time-dependent Volterra integro-differential equations. Their investigation specifically dealt with integro-differential equations involving two integration terms of the unknown function, both in terms of direct problem formulation and inverse problem resolution. In the research conducted by Jacob (2020) [8], the focus was on investigating various analytical and numerical approaches for tackling the Fredholm integral equation of the second kind. The methods explored included degenerated kernel methods, numerical methods, the projection method, the Nystrom method, and spectra methods.

By studying and analyzing these techniques, the author aimed to provide a comprehensive understanding of their applications in solving the aforementioned integral equation. In a separate study carried out by Nemati (2014) [11], Volterra-Fredholm integral equations were addressed, and a novel method based on shifted Legendre polynomials was introduced for approximating solutions. The unique characteristics of shifted Legendre polynomials were thoroughly discussed in this context. Furthermore, the properties of these polynomials, along with the shifted Gauss-Legendre nodes, were effectively utilized to reduce the Volterra-Fredholm integral equations to a more manageable matrix equation for a streamlined solution process. Yousef and Lin (2012) [14] devised a method to solve Volterra integral equations of the first kind and Fredholm-Volterra integral equations using the scaling function interpolation method. They provided a convergence theorem for the numerical solutions of these integral equations, ensuring the accuracy and reliability of their approach. Dastjerdi and Ghaini (2012) [7] explored a numerical solution for Volterra-Fredholm integral equations of the second kind. Their method involved the moving least squares technique and Chebyshev polynomials, enabling effective approximations of solutions. Abdou, El-Kalla and Al-Bugami (2011) [1] discussed the existence of a unique solution to the Volterra-Fredholm integral equation of the second kind. The Volterra integral term was considered in time using a continuous kernel, while the Fredholm integral term was dealt with in position using a numerical technique and a generalized singular kernel. The researchers reduced the V-FIESK to a system of Fredholm integral equations (SFIEs) using the Toeplitz matrix method and the Product Nystrom method, which led to obtaining a linear algebraic system of equations.
Subsequently, Muna and Iman (2018) [10] proposed novel algorithms for solving linear Volterra-Fredholm integral equations (LVFIEs) of the second kind. Their methods were founded on the trapezoidal rule, Weddle's rule, and Richardson's extrapolation, offering alternative approaches for addressing these types of integral equations. Merve and Ercan (2021) [9] introduced the Volterra-Fredholm integral equation using Hosoya polynomials. The outcomes of these methods were compared through figures, tables, and error analyses. The remarkable contributions of the aforementioned researchers served as inspiration for our work to propose an algorithm based on power series and shifted Chebyshev polynomials for solving Volterra and Fredholm integral equations**.** Our objective is to design an efficient and accurate approach that reduces computational effort while providing an approximate solution for linear Volterra-Fredholm integral equations (LVFIEs).

Definition of Basic Terms

In this section, we define some basic terms that would be encountered in this work.

Integral equations (Wazwaz, 2011) [16]

An integral equation is characterized by the presence of the unknown function $\xi(z)$ that needs to be determined within an integral expression. Typically, such an equation takes the following form:

$$\xi(z) = f(z) + \lambda \int_{h(z)}^{i(z)} K(z,t)\xi(t)dt \quad (1)$$

Where $\xi(z)$ is to be determined, $f(z)$ is a given function, λ is a free parameter, $K(z,t)$ is called the kernel of the integral equation, and $i(z)$ and $h(z)$ are the limits of integration

Collocation method *(*Adebisi *et al*., 2021) [2]

Collocation method is a method for the numerical solution of ordinary differential equations, partial differential equations and integral equations. The idea is to choose a finite-dimensional space of candidate solutions usually polynomials up to a certain degree and a seveal points in the domain called collocation points, and to select the solution which satisfies the given equation at the collocation point.

Volterra integral equations (Wazwaz, 2015) [16]

The Volterra integral equation is defined by having a constant lower limit and a variable upper limit. These equations are categorized into two groups: the first kind and the second kind. A linear Volterra equation of the first kind is expressed as follows:

$$0 = f(z) + \int_a^z K(z,t)\xi(t)dt \quad (2)$$

Approximate solution (Ayinde *et al*. 2021a) [6]

This is the expression obtained after the unknown constants have been found and substituted back into the assumed solution. It is referred to as approximate solution.
In this work, approximate solution used is given as

$$\xi_m(z) = \sum_{r=0}^{M} a_r H_r(z) \quad (3)$$

where $a_r, r = 0,1,2,3,4 \dots m$ are unknown constant to be found, $H_r(z)(r \geq 0)$ is the approximating polynomials of any kind. M is the degree of approximant, where in most cases the better approximate solution (i.e. close to the exact solution)

Power series: *given as*

$$\xi(z) = \sum_{r=0}^{M} c_r p^r(z) \quad (4)$$

where c_r are the constants coefficients. The power series is a polynomial function if all but a finite number are zero, but if most of c_r are nonzero, then its convergence is considered.

Chebyshev polynomials (Ayinde *et al*. 2021b) [7]

$$\xi(z) = \sum_{r=0}^{M} c_r T_r(z) \quad (5)$$

where c_r denotes the unknown constants to be calculated and $T_r(z)$ denotes the Chebyshev polynomials.
The Chebyshev polynomial denoted by $T_r(x)$ and valid in the interval $a \leq z \leq b$ is defined as

$$T_r(z) = \cos\left[rcos^{-1}\left(\frac{2z-(a+b)}{b-a}\right)\right] \quad , \; r = 0,1,2, \tag{6}$$

where the recurrence relationship is denoted as

$$T_{r+1}(z) = 2\left[\frac{2z-(a-b)}{b-a}\right]T_r(z) - T_{r-1}(z), \; a \leq z \leq b \tag{7}$$

To transform Chebyshev polynomials from the interval [a, b] to the interval [0, 1]
To go from interval [a, b] to interval [0, 1], the following procedures is carefully followed.
Here, we put a = 0 and b =1 in equation (10), we obtain

$$T_r(z) = \cos\left[rcos^{-1}\left(\frac{(2z-1)}{1-0}\right)\right] \tag{8}$$

$$T_r(z) = \cos[rcos^{-1}(2z-1)], \quad [0,1] \tag{9}$$

and also, from equation (11) we obtain

$$T_{r+1}(z) = 2(2z-1)T_r(z) - T_{r-1}(z) \quad r \geq 1 \tag{10}$$

As a result, for various values of r, we have the shifted Chebyshev polynomials shown below.
when

$$\left.\begin{array}{l} r = 0: T_0^*(z) = 1 \\ r = 1: T_1^*(z) = 2z-1 \\ r = 2: T_2^*(z) = 8z^2 - 8z + 1 \\ r = 3: T_3^*(z) = 32z^3 - 48z^2 + 18z - 1 \\ r = 4: T_4^*(z) = 128z^4 - 256z^3 + 160z^2 - 32z + 1 \\ r = 5: T_5^*(z) = 512z^5 - 1280z^4 + 1120z^3 - 400z^2 + 50z - 1 \\ \vdots \end{array}\right\} \tag{11}$$

Hence, the trial solution is now given as

$$\xi_M(z) = \sum_{r=0}^{M} c_r \, T_r^*(z) \tag{12}$$

Problem considered and methodology
In this section, we have examined the Volterra-Fredholm integral equation, where we approach the problem by assuming an approximate solution of varying degrees using both Power series and shifted Chebyshev polynomial methods. The general form of the problem under consideration is as follows:

$$\xi(z) = f(z) + \lambda \int_{i(z)}^{h(z)} K(z,t)\xi(t)dt \tag{13}$$

where $\xi(z)$ is unknown functions to be determined, $f(z)$ is a known function, λ is free parameter, $h(z) \; and \; i(z)$ are the limits of integration, $K(z,t)$ is a continuous function called Kernel.

Chebyshev Gauss-Lobbato collocation point for Power series
In our approach to solve equation (13) using Chebyshev Gauss-Lobatto collocation points, we consider an approximate solution in the following form:

$$\xi(z) = \sum_{r=o}^{M} c_r p^r(z) \tag{14}$$

Therefore, equation (14) can be rewritten as

$$\sum_{r=0}^{M} c_r p^r(z) = f(z) + \lambda \int_{i(z)}^{h(z)} K(z,t) \sum_{r=0}^{M} c_r p^r(t)\, d(t) \tag{15}$$

Where M is the degree of approximant, c_r are the unknown constants that need to be determined, and $p^r(z)$ represents the Power series defined by equation (4).
Therefore, equation (15) implies

$$c_0 p^0 + c_1 p(x) + c_2 p^2(x) + \cdots c_n p^n(z) = f(z)$$

$$+\lambda \int_{i(z)}^{h(z)} K(z,t)[c_0 + c_1 p^1(t) + c_2 p^2(t) + \cdots c_n p^n(t)]dt \tag{16}$$

Evaluating the integral part of equation (16) gives

$$c_0 + c_1 p(z) + c_2 p^2(z) + \cdots c_n p^n(z) = f(z) + \lambda I(z) \tag{17}$$

Where,

$$I(z) = \int_{i(x)}^{h(x)} k(x,t)[c_0 + c_1 p(t) + c_2 p^2(t) + \cdots c_n p^n(t)]\, dt \tag{18}$$

After evaluating the integral part of equation (17), the resulting equation is collocated at $z = z_k$ using the Chebyshev Gauss-Lobbato collocation point.

$$c_0 + c_1 p(z_k) + c_2 p^2(z_k) + \cdots c_n p^n(z_k) = f(z_k) + \lambda I(z_k) \tag{19}$$

Where,

$$z_k = cos\left(\frac{\pi k}{M}\right), \quad k \geq 0 \tag{20}$$

Thus, equation (20) is then put into matrix form as

$$PC = Q(z) \tag{21}$$

Where,

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots p_{1M} \\ p_{21} & p_{22} & \cdots p_{2M} \\ \vdots & \vdots & \ddots \; \vdots \\ p_{M1} & p_{M2} & \cdots p_{MM} \end{bmatrix} = (p_{i,j}), \; C = [c_0 \; c_1 \; c_2 \; \ldots c_M]^T, \; Q = [q_0 \; q_1 \; q_2 \; \ldots q_M]^T$$

Thus, equation (19) gives rise to $(M+1)$ algebraic linear equation in $(M+1)$ unknown constants and these equations are then solved via Maple 18 code to obtain the $(M+1)$ unknown constants and substitutable back into equation (4).

Chebyshev Gauss-Lobbato collocation point via shifted Chebyshev polynomials

Here, we assume an approximate solution of the form

$$\xi(z) = \sum_{r=o}^{M} c_r T_r^*(z) \tag{22}$$

Thus, substituting (22) into equation (15), we have

$$\sum_{r=0}^{M} c_r T_r^*(z) = f(z) + \lambda \int_{i(z)}^{h(z)} K(z,t)\left(\sum_{r=0}^{M} c_r T_r^*(t)\right) d(t) \tag{23}$$

Thus, equation (23) implies

$$c_0 T_0^*(z) + c_1 T_1^*(z) + c_2 T_2^*(z) + \cdots c_m T_m^*(z) = f(z)$$

$$+\lambda \int_{i(x)}^{h(x)} K(z,t)(T_0^*(t) + c_1 T_1^*(t) + c_2 T_2^*(t) + \cdots c_m T_m^*(t))dt \tag{24}$$

By evaluating equation (24)'s integral part, we obtain

$$c_0 T_0^*(z) + c_1 T_1^*(z) + c_2 T_2^*(z) + \cdots c_m T_m^*(z) = f(z) + \lambda G(z) \tag{25}$$

Where

$$G(z) = \int_{i(z)}^{h(z)} K(z,t)[c_0 T_0^*(t) + c_1 T_1^*(t) + c_2 T_2^*(t) + \cdots c_m T_m^*(t)]\, dt$$

As a result, equation (25) is evaluated integrally, and the resulting equation is collocated at $z = z_k$ using the Chebyshev Gauss-Lobbato collocation point.

$$c_0 T_0^*(z_k) + c_1 T_1^*(z_k) + c_2 T_2^*(z_k) + \cdots c_m T_m^*(z_k) = f(z_k) + \lambda G(z_k) \tag{26}$$

Where

$$z_k = \cos\left(\frac{\pi k}{M}\right), k \geq 0 \tag{27}$$

Thus, equation (26) is then put into matrix form as

$$GC = F(z) \tag{28}$$

Where,

$$G = \begin{bmatrix} g_{11} & g_{12} & \cdots g_{1M} \\ g_{21} & g_{22} & \cdots g_{2M} \\ \vdots & \vdots & \ddots \vdots \\ g_{M1} & g_{M2} & \cdots g_{MM} \end{bmatrix} (g_{i,j}),\ C = [c_0\ c_1\ c_2 \ldots c_M]^T,\ F = [f_0\ f_1\ f_2 \ldots f_M]^T$$

Thus, equation (26) gives rise to $(M+1)$ algebraic linear equation in $(M+1)$ unknown constants, and these equations are then solve via Maple 18 code to obtain the $(M+1)$ unknown constants $c_r(r = 0,1,\ldots M)$, the approximate solution is obtained by substituting these values back into equation (22)

Numerical Examples

To evaluate the method's efficacy and clarity, we provided three numerical examples in this section. A MAPLE 18 program was used to perform the calculations. Let $\xi_m(z)$ and $\xi(z)$ stand for the approximate and exact solutions, respectively. $\text{Error}_M = |\xi_m(z) - \xi(z)|$

Numerical Example 1

Let's consider the Fredholm integral equation of the second kind (Wazwaz, 2015) [16]

$$\xi(z) - \frac{2}{3}z - \int_0^1 zt\xi(t)dt \tag{29}$$

with exact solution

$$\xi(z) = z \tag{30}$$

Numerical Example 2

Let's consider the Volterra integral equation of the second kind (Wazwaz, 2015) [16]

$$\xi(z) - z - \int_0^z (z-t)\xi(t)dt \tag{31}$$

with exact solution

$$\xi(z) = \sin(z) \tag{32}$$

Numerical Example 3

Let's consider the Volterra integral equation of the second kind (Wazwaz, 2011) [15]

$$\xi(z) - 6z + z^3 - \frac{1}{2}\int_0^z t\xi(t)dt \tag{33}$$

with exact solution

$$\xi(z) = \sin(z) \tag{34}$$

Results & Discussion

In this section, we utilized Chebyshev-Gauss-Lobbatto collocation points to solve sample problems. The numerical solutions obtained through this method were compared with the exact solutions of the sample problems. Moreover, we also compared the absolute errors of the results obtained using power series and shifted Chebyshev polynomials as basis functions. In the tables, the following notations were used to present the results.

PS: Solution via Chebyshev Gauss-Lobatto collocation using Power series.

SCP: Solution via shifted Chebyshev polynomials, which are the proposed method in this paper.

Table 1: Results and Absolute errors obtained for example 1 using PS & SCP for case M = 3

z	ξ_3(Exact)	ξ_{PS}(Approx.)	ξ_{SCP}(Approx.)	$error_{PS}$	$error_{SCP}$
0.0	0.000000000	0.000000000	1.4875e-10	0.0000000	1.4875e-10
0.2	0.200000000	0.200000000	0.20000000	0.0000000	0.000000
0.4	0.400000000	0.400000000	0.39999991	0.0000000	1.000e-10
0.6	0.600000000	0.600000000	0.59999991	0.0000000	1.000e-10
0.8	0.800000000	0.800000000	0.79999991	0.0000000	1.000e-10
1.0	1.000000000	1.000000000	0.99999991	0.0000000	1.000e-10

Table 2: Results and Absolute errors obtained for example 1 using PS & SCP for case M = 5

z	ξ_5(Exact)	ξ_{PS}(Approx.)	ξ_{SCP}(Approx.)	$error_{PS}$	$error_{SCP}$
0.0	0.000000000	0.000000000	1.7769x 10^{-10}	0.0000000	1.7769e-10
0.2	0.200000000	0.200000000	0.199999991	0.0000000	0.0000e-10
0.4	0.400000000	0.400000000	0.399999991	0.0000000	2.0000e-10
0.6	0.600000000	0.600000000	0.599999991	0.0000000	4.0000e-10
0.8	0.800000000	0.800000000	0.799999991	0.0000000	5.0000e-10
1.0	1.000000000	1.000000000	0.999999991	0.0000000	3.0000e-10

Table 3: Results and Absolute errors obtained for example 2 using PS & SCP for case M = 3

z	ξ_3(Exact)	ξ_{PS}(Approx.)	ξ_{SCP}(Approx.)	$error_{PS}$	$error_{SCP}$
0.0	0.000000000	0.0000000	0.73680e-10	0.0000000	1.7360e-10
0.2	0.198669331	0.1986681	0.198668098	1.2324e-06	1.2324e-06
0.4	0.389418342	0.3894204	0.389420441	2.0990e-06	2.0990e-06
0.6	0.564642473	0.5646482	0.564648185	5.7112e-06	5.7113e-06
0.8	0.717356090	0.7173563	0.717356307	2.1620e-07	2.1630e-07
1.0	0.841470985	0.8414705	0.841470522	4.6330e-07	4.6320e-07

Table 4: Results and Absolute errors obtained for example 2 using PS & SCP for case M = 5

z	ξ_5(Exact)	ξ_{PS}(Approx.)	ξ_{SCP}(Approx.)	$error_{PS}$	$error_{SCP}$
0.0	0.000000000	0.0000000000	0.73680e-10	0.0000000	1.7360e-10
0.2	0.198669330	0.1986680984	0.198668098	1.2324e-06	1.2324e-06
0.4	0.389418342	0.3894204413	0.389420441	2.0990e-06	2.0990e-06
0.6	0.564642473	0.5646481846	0.564648185	5.7112e-06	5.7113e-06
0.8	0.717356090	0.7173563071	0.717356307	2.1620e-07	2.1630e-07
1.0	0.841470984	0.8414705215	0.841470522	4.6330e-07	4.6320e-07

Table 5: Results and Absolute errors obtained for example 2 using PS & SCP for case M = 3

z	ξ_3(Exact)	ξ_{PS}(Approx.)	ξ_{SCP}(Approx.)	$error_{PS}$	$error_{SCP}$
0.0	0.000000000	0.000000000	-1.9349691e-10	0.0000000	1.93497e-10
0.2	1.200000000	1.199999991	1.200000000	9.0000e-09	0.0000000
0.4	2.400000000	2.399999991	2.400000000	9.0000e-09	0.0000000
0.6	3.600000000	3.599999991	3.600000000	9.0000e-09	0.0000000
0.8	4.800000000	4.799999991	4.300000000	9.0000e-09	0.0000000
1.0	6.000000000	5.999999991	5.999999991	9.0000e-09	0.0000000

Table 6: Results and Absolute errors obtained for example 2 using PS & SCP for case M = 5

z	ξ_5(Exact)	ξ_{PS}(Approx.)	ξ_{SCP}(Approx.)	$error_{PS}$	$error_{SCP}$
0.0	0.000000000	0.000000000	-6.4812425e-10	0.0000000	6.48120e-10
0.2	1.200000000	1.200000000	1.200000000	0.0000000	0.0000000
0.4	2.400000000	2.400000000	2.400000000	0.0000000	0.0000000
0.6	3.600000000	3.600000000	3.600000000	0.0000000	0.0000000
0.8	4.800000000	4.800000000	4.799999991	0.0000000	9.0000e-09
1.0	6.000000000	6.000000000	5.999999991	0.0000000	0.0000000

Conclusions

Table's 1-6 display the numerical solutions obtained using the Volterra-Fredholm integral equations (VFIEs) with both Power series and shifted Chebyshev polynomials as basis functions. We compared the approximate solutions and absolute errors resulting from these two methods. It was observed that, for varying degrees of M, an improvement in the approximation of the exact solution occurred in the final iterations of all the problems considered (as evident from the error tables). Additionally, we noticed that in some cases, the Power series outperformed the shifted Chebyshev polynomials as a basis function, yielding more accurate results. This finding indicates that the choice of basis function can significantly impact the accuracy of the numerical solutions obtained using the VFIEs.

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