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# A reconstructive variation AL iteration technique for solving tenth-order boundary value problems using the second kind chebyshev polynomials

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**Abstract**
The reconstructive variational iteration technique using Chebyshev polynomials of the second kind was used to obtain the numerical solution of tenth order boundary value problems (BVPs.) in this paper. The proposed modification is made by constructing Chebyshev polynomials of the second kind for the given boundary value problems (BVPs.) and used as a basis functions for the approximation. Numerical examples where also given to show the efficiency and reliability of the proposed method. Calculations were performed using maple 18 software.

**Keywords:** Approximate solutions, boundary value problems, chebyshev polynomials of the second kind, reconstructive variation al iteration technique

## Introduction

The variational iteration method (VIM) is widely known as an analytical technique that has been used to solve various forms of differential and Integro-differential equations with the appropriate correctional function. The choice of the Lagrange multiplier determines, to an extent, the accuracy of the results. We consider the boundary value problem of the form:

$$\mu_{10}\frac{d^{10}}{dz^{10}}w + \mu_9\frac{d^9}{dz^9}w + \mu_8\frac{d^8}{dz^8}w \;\dots\; \mu_1\frac{d}{dz}w + \mu_0 w = g(z), \quad a < z < b, \tag{1}$$

With boundary conditions

$$\begin{aligned} &w(a) = A_1\,,\; w'(a) = A_2,\, w''(a) = A_3 \;\dots\; w^{(9)}(a) = A_{10} \\ &w(b) = B_1, w'(b) = B_2,\, w''(b) = B_3 \;\dots\; w^{(9)}(b) = B_{10} \end{aligned} \tag{2}$$

Where $\mu_{10}, \mu_9, \mu_8 \cdots \mu_0$ are constants with $g(z)$ continuous on [$a$, $b$]. This type of problems are relevant in mathematical modeling of real life situations such as viscoelastic flow, heat transfer, and in other fields of engineering sciences. Over the years, several numerical techniques have been developed for solving problems of this kind. (Kasi Viswanadham and Sreenivasulu, 2015) [5] used Galerkin Method with Septic B-splines for the solution of tenth order boundary value problems. (Ali *et al.*, 2018) [3] Used reproducing kernel Hilbert space method for solving tenth order boundary value problems. Michael *et al.* 2018) [14] successfully solved first and second order differential equations (ODE) with first kind Chebychev polynomials as basic functions using Variational Iteration method. The paper presented satisfactory results with excellent convergence to the exact solution of first and second order ordinary differential equations in the literature. Maleknejad and Tamamgar (2014) [12] applied a new reconstruction of Variational iteration method on nonlinear Volterra integro differential equations. In their paper, a parametric Iteration method was constructed and used to solve nonlinear Volterra integro-differential equations which made the problem not only easier to solve but to produce good results that converged rapidly to the exact solution of the problem. Ojobor (2018) [10] investigated the convergence of the method as applied to second order boundary value problems (BVPs) at the various collocation points.

Irandoust-pakchin (2011) [9] constructed a modification of the He's variational iteration method (MVIM) to solve fractional ordinary differential equations and fractional partial differential equations with the aim of obtaining the exact solutions. The method is said to be computationally efficient to apply to several fractional ordinary differential and fractional partial differential equations with few iterations yielding high convergence to the exact solutions. (Mohammed and Kacem, 2019) [8] used Variational iteration method (VIM) by adopting Aboodh transformation method. In the paper, the author transformed the linear and nonlinear ordinary differential equations into an algebraic form, in order to be able to identify the Lagrange multiplier straightforward. At the end, the method is found to be efficient and easy to solve differential equations. The results actually shows that the method is efficient and simple. Austine *et al.* (2022) [3] introduced a new iteration method called the A** iteration method for the solution of a fractional Volterra–Fredholm integro-differential equations. According to the authors, they establish some weak and strong convergence results of the method for fixed points of generalized *a*-non expansive mappings in the appropriate spaces. The results obtained by the authors actually improved and generalized several well-known results in previous works. (Tunde *et al.*, 2013) [11] Solved linear and nonlinear thirteenth order differential equations in boundary value problems via variational iteration method (VIM) to find approximate solutions. The described the method as extremely simple to be used because of the accuracy of the results of the solution obtained and the efficiency of the method. It concluded that the method is a power-full tool to search for solutions of various linear and nonlinear boundary value problems. Benmezai and Sedkaoui, (2021) [4] investigated the existence of a positive solution to the third-order boundary value problem. In this paper, the variational iteration technique is used to solve the boundary value problem of the tenth order using Chebyshev polynomials of the second kind. The correction function is corrected for the BVP in this proposed method, and the Lagrange multiplier is optimally computed using variational theory. The proposed method works well, and the results so far are encouraging and consistent. Finally, the solution is given in an infinite series, which is usually convergent to an exact solution.

**Standard variation AL Iteration Technique**

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lw + N_l w - g(z) = 0, \tag{3}$$

Where $L$ is a linear operator, $N_l$ a nonlinear operator and $g(x)$ is the inhomogeneous term According to variational iteration method, we can construct a correction functional as follows:

$$w_{r+1} = w_r(z) + \int_0^z \lambda(\tau)\left(Lw_r(\tau) + N_l\widetilde{w_\iota(\tau)} - g(\tau)\right)d\tau \tag{4}$$

Where $\lambda(\tau)$ is a Lagrange multiplier, which can be identified optimally via Variational iteration technique? The subscripts $n$ denote the nth approximation, $\widetilde{w_\iota}$ is considered as a restricted variation. i.e., $\widetilde{w_\iota}$= 0. The relation (4) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. In this method, we need to determine the Lagrange multiplier $\lambda(\tau)$ optimally and hence the successive approximation of solution $v$ will be readily obtained upon using the Lagrange multiplier and our $w_0$, and the solution is given by

$$\lim_{r-\infty} w_r(z) = w(z) \tag{5}$$

The Lagrange Multiplier also play an important role in determining the solution of the problem, and can be defined as follows:

$$\lambda(\tau) = (-1)^k \frac{1}{(k-1)!}(\tau - z)^{k-1} \tag{6}$$

**Chebyshev Polynomials of the Second Kind**

The Chebyshev Polynomials of the second kind are orthogonal polynomials with respect to the weight function

$$w(z) = (1 - z^2)^{\frac{1}{2}} \,\forall\, z \in [-1,1] \tag{7}$$

The Chebyshev polynomials of the second kind is defined by

$$T_k(z) = \frac{\sin[(k+1)cos^{-1}z]}{sin(cos^{-1}z)} \tag{8}$$

Hence, the first five Chebyshev polynomials of the second kind is given below:

$$\left.\begin{array}{ll} k=0: T_0(z)= & 1 \\ k=1: T_1(z)= & 2z \\ k=2: T_2(z)= & 4z^2-1 \\ k=3: T_3(z)= & 8z^3-4z \\ k=4: T_4(z)= & 16z^4-12z^2+1 \\ k=5: T_5(z)= & 32z^5-32z^3+6z \\ & \vdots \end{array}\right\} \tag{9}$$

**Shifted Chebyshev Polynomials of the Second Kind**

The shifted Chebyshev polynomials of the second kind are orthogonal polynomials with respect to the weight function:

$v^*(z)=\sqrt{z(1-z)}\ \forall\ z\in[0,1]$ with starting values $T^*{}_0(z)=1$ and $T^*{}_1(z)=4z-2.$

This is defined by the relation

$$T_k{}^*(x)=T_k(2z-1)\ \forall\ z\in[0,1]. \tag{10}$$

Hence, the first five shifted Chebyshev polynomials of the second kind is given below

$$\left.\begin{array}{ll} k=0: T^*{}_0(z)= & 1 \\ k=1: T^*{}_1(z)= & 2z-2 \\ k=2: T^*{}_2(z)= & 16z^2-16z+3 \\ k=3: T^*{}_3(z)= & 64z^3-96z^2+40z-4 \\ k=4: T^*{}_4(z)= & 256z^4-512z^2+336z^3-80z+5 \\ k=5: T^*{}_5(z)= & 1024z^5-2560z^3+2304z^3=896z^3+140z-6 \\ & \vdots \end{array}\right\} \tag{11}$$

**Modified Variational Iteration Technique Using Chebyshev and Shifted Chebyshev Polynomials of the Second Kind (MVITCP-SCP)**

Using (3) and (4), we assume an approximate solution of the form:

$$w_{r,N-1}(z)=\sum_{r=0}^{N-1}\mu_{r,N-1}T_{r,N-1}(z) \tag{12}$$

$$w^*{}_{r,N-1}(z)=\sum_{r=0}^{N-1}a_{r,N-1}T^*{}_{r,N-1}(z) \tag{13}$$

Where $T_{r,N-1}(z)$ and $T^*{}_{r,N-1}(z)$ are Chebyshev polynomials of the second kind and shifted Chebyshev polynomials of the second kind respectively, $\mu_{r,N-1}$are constants to be determined, and $N$ the degree of approximant. Hence we obtain the following iterative method

$$w_{r+1,N-1}(z)=\sum_{r=0}^{N-1}\mu_{r,N-1}T_{r,N-1}(z)\ +\int_0^z\lambda(\tau)\Big(L\sum_{r=0}^{N-1}\mu_{r,N-1}T_{r,N-1}(\tau)+ N_l\sum_{r=0}^{N-1}\mu_{r,N-1}T_{r,N-1}(\tau)-g(\tau)\Big)d\tau \tag{14}$$

$$w^*{}_{r+1,N-1}(z)=\sum_{r=0}^{N-1}\mu_{r,N-1}T^*{}_{r,N-1}(z)+\int_0^z\lambda(\tau)\Big(L\sum_{i=0}^{N-1}\mu_{r,N-1}T^*{}_{r,N-1}(\tau)+ N_l\sum_{r=0}^{N-1}\mu_{r,N-1}T^*{}_{r,N-1}(\tau)-g(\tau)\Big)d\tau \tag{15}$$

**Mathematical Application**

In this section we applied the proposed method to solve four examples. Numerical results also show the accuracy and efficiency of the proposed scheme.

**Example 1:** Considers the following tenth order boundary value problem **(**Kasi Viswanadham & Sreenivasulu, 2015**)** [5].

$$w^{(10)} + w = -10(2zsinz - 9cosz), \qquad -1 \le z \le 1$$
$$w(-1) = w(1) = 0,\ w'(1) = -v'(-1) = 2cos1,\ v''(-1) = v''(1) = 2cos1 - 4sin1 \qquad (16)$$

$$w'''(-1) = -w'''(1) = 6cos1 + 6sin1, w^{(4)}(-1) = w^{(4)}(1) = -12cos1 + 8sin1 \qquad (17)$$

The exact solution for the problem is

w=(z^2-1)cosz (18)

The correction functional for the boundary value problem (16) and (17) is given as

$$w_{r+1} = w_r(z) + \int_0^z \lambda(\tau)\left(w^{(10)} + w + 10(2\tau sin\tau - 9cos\tau)\right)d\tau \qquad (19)$$

Where, $\lambda(\tau) = \frac{(-1)^{10}(\tau-z)^9}{9!}$ is the Lagrange multiplier applying the reconstructive variation AL iteration technique using the Chebyshev polynomials of the second kind, we assume an approximate solution of the form

$$w(z) = \sum_{r=0}^{9} \mu_{r,9} T_{r,9}(z) \qquad (20)$$

Hence, we get the following iterative formula:

$$w_{r+1,N-1}(z) = \sum_{r=0}^{9} \mu_{r,9} T_{r,9}(z) + \int_0^z \frac{(\tau-z)^9}{9!}\left(\frac{d^{10}}{d\tau^{10}}\left(\sum_{r=0}^{9} \mu_{r,9} T_{r,9}(\tau)\right) + \sum_{r=0}^{9} \mu_{r,9} T_{r,9}(\tau)\, s + 10(2\tau sin\tau - 9cos\tau)\right)d\tau \qquad (21)$$

$$w_{r+1,N-1}(z) = \mu_{0,9}T_{0,9}(z) + \mu_{1,9}T_{1,9}(z) + \mu_{2,9}T_{2,9}(z) + \mu_{3,9}T_{3,9}(z) + \mu_{4,9}T_{4,9}(z) + \mu_{5,9}T_{5,9}(z) + \mu_{6,9}T_{6,9}(z) + \mu_{7,9}T_{7,9}(z) + \mu_{8,9}T_{8,9}(z) + \mu_{9,9}T_{9,9}(z) + \int_0^z \frac{(\tau-z)^9}{9!}\left(\frac{d^{10}}{d\tau^{10}}\left(\sum_{r=0}^{9} \mu_{r,9} T_{r,9}(\tau)\right) + \sum_{r=0}^{9} \mu_{r,9} T_{r,9}(\tau) + 10(2\tau sin\tau - 9cos\tau)\right)d\tau \qquad (22)$$

Hence: using (9), iterating and applying the boundary conditions (17) the values of the unknown constants can be determined as follows

$$\mu_{0,9} = -0.6894219292,\ \mu_{1,9} = 0,\ \mu_{2,9} = 0.2793375651,\ \mu_{3,9} = 0,\ \mu_{4,9} = -0.03060089596$$

$$\mu_{5,9} = 0,\ \mu_{6,9} = 0.000634087456,\ \mu_{7,9} = 0,\ \mu_{8,9} = -0.0000055222284,\ \mu_{9,9} = 0$$

Consequently, the series solution is given as

$$w(z) = -0.9999999999 + 4.79509653610^{-14}z^{18} - 1.15185403810^{-11}z^{16}$$

$$+2.09914644510^{-9}z^{14} - 2.777660867910^{-7}z^{12} + 0.00002507716049z^{10} - 0.00141369048z^{8} + 0.04305555574z^{6} - 0.5416666673z^{4} + 1.500000000z^{2} + \frac{1}{12164510040883200}z^{20} \qquad (23)$$

**Example 2:** Considers the following tenth order boundary value problem **(**Kasi Viswanadham & Sreenivasulu, 2015**)** [5]

$$w^{(10)} - (z^2 - 2z)w = 10cosz - (z-1)^3 sinz, \quad -1 \le z \le 1 \tag{24}$$

With conditions

$$w(-1) = 2sin1,\ w(1) = 0,\ w'(-1) = -2cos1 - sin1,\ w'(1) = sin1,$$
$$w''(-1) = 2cos1 - 2sin1, w''(1) = 2cos1, w'''(-1) = 2cos1 + 3sin1,$$
$$w'''(1) = -3sin1, w^{(4)}(-1) = -4cos1 + 2sin1, w^{(4)}(1) = -4cos1 \tag{25}$$

The exact solution for the problem is

$$w(z) = (z-1)sinz \tag{26}$$

The correction functional for the boundary value problem (24) and (25) is given as

$$w_{r+1} = w_r(z) + \int_0^z \lambda(\tau)\left(w^{(10)} - (\tau^2 - 2\tau)w - 10cos\tau + (\tau-1)^3 sin\tau\right)d\tau \tag{27}$$

Where, $\lambda(\tau) = \frac{(-1)^{10}(\tau-z)^9}{9!}$ is the Lagrange multiplier. Applying the reconstructive variation AL iteration technique using equation (9) by assume an approximate solution equation (20) to get the following iterative formula

$$w_{r+1,N-1}(z) = \sum_{r=0}^{9} \mu_{r,9} T_{r,9}(z) + \int_0^z \frac{(\tau-z)^9}{9!} \left(\frac{d^{10}}{d\tau^{10}}\left(\sum_{r=0}^{9} \mu_{r,9} T_{r,9}(\tau)\right) - (t^2 - 2\tau)\sum_{r=0}^{9} \mu_{r,9} T_{r,9}(\tau) - 10cos\tau + (\tau-1)^3 sin\tau\right) d\tau \tag{28}$$

$$w_{r+1,N-1}(z) = \mu_{0,9}T_{0,9}(z) + \mu_{1,9}T_{1,9}(z) + \mu_{2,9}T_{2,9}(z) + \mu_{3,9}T_{3,9}(z) + \mu_{4,9}T_{4,9}(z) + \mu_{5,9}T_{5,9}(z) + \mu_{6,9}T_{6,9}(z) + \mu_{7,9}T_{7,9}(z) + \mu_{8,9}T_{8,9}(z) + \mu_{9,9}T_{9,9}(z) + \int_0^z \frac{(\tau-z)^9}{9!} \left(\frac{d^{10}}{d\tau^{10}}\left(\sum_{r=0}^{9} \mu_{r,9} T_{r,9}(\tau)\right) - (\tau^2 - 2\tau)\sum_{r=0}^{9} \mu_{r,9} T_{r,9}(\tau) - 10cos\tau + (\tau-1)^3 sin\tau\right) d\tau \tag{29}$$

Hence: using (9), iterating and applying the boundary conditions (25) the values of the unknown constants can be determined as follows

$$\mu_{0,9} = 0.2298068577, \quad \mu_{1,9} = -0.4596139413, \mu_{2,9} = 0.2199001736, \mu_{3,9} = 0.01981310979$$

$$\mu_{4,9} = -0.0097811260, \mu_{5,9} = -0.00025126139, \mu_{6,9} = 0.000124782986$$

$$\mu_{7,9} = 0.000001507041, \quad \mu_{8,9} = -7.750495810^{-7}, \ \mu_{9,9} = -5.382310^{-9}$$

Consequently, the series solution is given as

$$w(z) = -\frac{1}{851515702861824000}z^{22} + \frac{1}{362880}z^{10} + 4.306042598\,10^{-18}z^{21}$$
$$-4.1102918\,10^{-18}z^{20} + 8.220673310^{-18}z^{19} + 2.8114568210^{-18}z^{20}$$
$$-2.811455710^{-15}z^{17} - 7.6471638210^{-13}z^{16} + 7.647164510^{-13}z^{15}$$
$$-1.60590438010^{-10}z^{14} - 1.60590438810^{-10}z^{13} - 2.50521083810^{-8}z^{12}$$
$$+2.50521083910^{-8}x^{11} - 0.000002755737600z^{9} - 0.0001984126925z^{8}$$
$$+0.0001984127232z^{7} + 0.008333333316z^{6} - 0.008333333258z^{6}$$
$$-0.166666666z^{4} + 0.1666666673z^{3} + 1.000000000z^{2}$$
$$-1.000000000z + 1.\,10^{-10} \tag{30}$$

**Example 3:** Considers the following tenth order boundary value problem **(**Kasi Viswanadham & Sreenivasulu, 2015**)** [5]

$$w^{(10)} - w'' + wz = (-8 + z - z^2)e^{z}, \qquad 0 \le z \le 1 \tag{31}$$

$$w(0) = 1,\ w(1) = 0,\ w'(0) = 0,\ w'(1) = -e,\ w''(0) = -1,\ w''(1) = -2e,\ w'''(0) = -2,$$
$$w'''(1) = -3e,\ w^{(4)}(0) = -3,\ w^{(4)}(1) = -4e \tag{32}$$

The exact solution for the problem is

$$w(z) = (1 - z)e^{z} \tag{33}$$

The correction functional for the boundary value problem (31) and (32) is given as

$$w_{r+1} = w_r(z) + \int_0^z \lambda(\tau)\left(w^{(10)} - w'' + w\tau - (-8 + \tau - \tau^2)e^{\tau}\right)d\tau \tag{34}$$

Applying the reconstructive variational iteration technique using the shifted Chebyshev polynomials of the second kind, we assume an approximate solution of the form

$$w^*{}_{r,9}(z) = \sum_{r=0}^{9} \mu_{r,9} T^*{}_{r,9}(z) \tag{35}$$

Hence, we get the following iterative formula:

$$w^*{}_{r+1,N-1}(z) = \sum_{r=0}^{9} \mu_{r,9} T^*{}_{r,9}(z) + \int_0^z \frac{(\tau - z)^9}{9!}\left(\frac{d^{10}}{d\tau^{10}}\left(\sum_{r=0}^{9} \mu_{r,9} T^*{}_{r,9}(\tau)\right) - \frac{d^2}{d\tau^2}\left(\sum_{r=0}^{9} \mu_{r,9} T^*{}_{r,9}(\tau)\right) + \tau\left(\sum_{r=0}^{9} \mu_{r,9} T^*{}_{r,9}(\tau)\right) - (-8 + \tau - \tau^2)e^{\tau}\right)d\tau \tag{36}$$

$$w^*{}_{r+1,N-1}(z) = \mu_{0,9}T^*{}_{0,9}(z) + \mu_{1,9}T^*{}_{1,9}(z) + \mu_{2,9}T^*{}_{2,9}(z) + \mu_{3,9}T^*{}_{3,9}(z) + \mu_{4,9}T^*{}_{4,9}(z) + \mu_{5,9}T^*{}_{5,9}(z) + \mu_{6,9}T^*{}_{6,9}(z) + \mu_{7,9}T^*{}_{7,9}(z) + \mu_{8,9}T^*{}_{8,9}(z) + \mu_{9,9}T^*{}_{9,9}(z) + \int_0^z \frac{(\tau - z)^9}{9!}\left(\frac{d^{10}}{d\tau^{10}}\left(\sum_{r=0}^{9} \mu_{r,9} T^*{}_{r,9}(\tau)\right) - \frac{d^2}{d\tau^2}\left(\sum_{r=0}^{9} a_{r,9} T^*{}_{r,9}(\tau)\right) + \tau\left(\sum_{r=0}^{9} \mu_{r,9} T^*{}_{r,9}(\tau)\right) - (-8 + \tau - \tau^2)e^{\tau}\right)d\tau \tag{37}$$

Hence: using (10), iterating and applying the boundary conditions (32) the values of the unknown constants can be determined as follows

$\mu_{0,9} = 0.85882297,\ \mu_{1,9} = -0.08867370\ ,\ \mu_{2,9} = -0.14943302, \mu_{3,9} = -0.04596561$
$\mu_{4,9} = -0.00836859,\ \mu_{5,9} = -0.00109864,\ \mu_{6,9} = -0.00011325$
$\mu_{7,9} == 0.000009646, \mu_{8,9} = -6.7910^{-7}, \mu_{9,9} = -4.3010^{-8}$

Consequently, the series solution is given as

$$\begin{aligned} w^*(z) &= 5.18535090310^{-16}z^{19} - 1.24840417710^{-15}z^{18} - 2.2470217110^{-14}z^{17} \\ &-7.1733148710^{-13}z^{16} - 1.07066855310^{-11}z^{15} - 1.49114473110^{-14}z^{14} \\ &-1.9208562010^{-9}z^{13} - 2.2296445247110^{-8}z^{12} - 2.50521063110^{-7}z^{11} \\ &-0.000002480158643z^{10} - 0.001190656000z^7 - 0.006943808000z^6 \\ &-0.03333334400z^5 - 0.1250004000z^4 - 0.3333332000z^3 - 0.4999998400z^2 \\ &-6.20000000\ 10^{-8}z - 0.00002201600000z^9 - 0.0001738240000z^8 \\ &+3.283801013\ 10^{-17}z^{20} + 0.9999999710 \end{aligned} \tag{38}$$

**Example 4:** Considers the following tenth order boundary value problem (Ali *et al.*, 2018) [2]

$$w^{(10)} = -(80 + 19z + z^2)e^z, \quad 0 \le z \le 1 \tag{39}$$

With boundary conditions

$$w(0) = 0,\ w(1) = 0,\ w''(0) = 0,\ w''(1) = -4e,\ w^{(4)}(0) = -8, w^{(4)}(1) = -16e,$$

$$w^{(6)}(0) = -24,\ w^{(6)}(1) = -36e,\ w^{(8)}(0) = -48,\ w^{(8)}(1) = -64e. \tag{40}$$

The exact solution of the example is

$$w(z) = z(1-z)e^z. \tag{41}$$

The correct functional for the boundary value problem (39) and (40) is given as

$$w_{r+1} = w_r(z) + \int_0^z \lambda(\tau)\left(w^{(10)} + (80 + 19\tau + \tau^2)e^\tau\right)d\tau \tag{42}$$

Where $\lambda(\tau) = \frac{(-1)^{10}(\tau-z)^9}{9!}$ is the Lagrange multiplier?

Applying the reconstructive variation AL iteration technique using equation (11) by assume an approximate solution equation (35) to get the following iterative formula

$$w^*_{r+1,N-1}(z) = \sum_{r=0}^{9}\mu_{r,9}T^*_{r,9}(z) + \int_0^z \frac{(\tau-z)^9}{9!}\left(\frac{d^{10}}{d\tau^{10}}\left(\sum_{r=0}^{9}\mu_{r,9}T^*_{r,9}(\tau)\right) + (80 + 19\tau + \tau^2)e^\tau\right)d\tau \tag{43}$$

$$\begin{aligned} w^*_{r+1,N-1}(z) &= \mu_{0,9}T^*_{0,9}(z) + \mu_{1,9}T^*_{1,9}(z) + \mu_{2,9}T^*_{2,9}(z) + \mu_{3,9}T^*_{3,9}(z) + \mu_{4,9}T^*_{4,9}(z) + \\ &\mu_{5,9}T^*_{5,9}(z) + \mu_{6,9}T^*_{6,9}(z) + \mu_{7,9}T^*_{7,9}(z) + \mu_{8,9}T^*_{8,9}(z) + \mu_{9,9}T^*_{9,9}(z) + \\ &\int_0^z \frac{(\tau-z)^9}{9!}\left(\frac{d^{10}}{d\tau^{10}}\left(\sum_{r=0}^{9}\mu_{r,9}T^*_{r,9}(\tau)\right) + (80 + 19\tau + \tau^2)e^\tau\right)d\tau \end{aligned} \tag{44}$$

Hence: using (10), iterating and applying the boundary conditions (40) the values of the unknown constants can be determined as follows

$\mu_{0,9} = -0.04433593750, \mu_{1,9} = 0.3546949487, \mu_{2,9} = -0.0673177833, \mu_{3,9} = -0.07890081769$

$\mu_{4,9} = -0.02353050595, \mu_{5,9} = -0.004240927095, \mu_{6,9} = -0.000553385167$

$\mu_{7,9} = -0.00005696608660, \mu_{8,9} = -0.000004650297619, \mu_{9,9} = -3.390857798\ 10^{-7}$

Consequently, the series solution is given as

$$w^*(z) = -\frac{1}{114328101888000}z^{18} - \frac{1}{2032499589120}z^{17} - \frac{1}{93405312000}z^{16} - \frac{1}{6706022400}z^{15} - \frac{1}{518918400}z^{14} - \frac{1}{43545600}z^{13} - \frac{1}{3991680}z^{12} - \frac{1}{403200}z^{11} - \frac{1}{45360}z^{10} - 0.00017361193z^9 - 0.001190476190z^8 - 0.006944435246z^7 - 0.03333333334z^6 - 0.1250000440z^5 - 0.3333333333z^4 - 0.499999977z^3 + 0.9999999435z \quad (45)$$

**Tables**

**Table 1:** The result of the proposed method compared Galerkin Method with Septic B-splines (Kasi Viswanadham and Sreenivasulu Ballem, 2015) [5]

| Z | Exact solution | Approximate solution | Absolute Error by the proposed method | GMSB-s Error |
|---|---|---|---|---|
| -0.8 | -0.2508144153 | - 0.2508144150 | 3.000000e-10 | 5.483627e-06 |
| -0.6 | -0.5282147935 | - 0.5282147940 | 5.000000e-10 | 9.536743e-07 |
| -0.4 | -0.7736912350 | -0.7736912350 | 0.000000e-00 | 8.702278e-06 |
| -0.2 | -0.9408639147 | -0.9408639150 | 3.000000e-10 | 2.980232e-07 |
| 0.0 | -1.0000000000 | -0.9999999999 | 1.000000e-10 | 1.955032e-05 |
| 0.2 | -0.9408639147 | -0.9408639150 | 3.000000e-10 | 2.920628e-05 |
| 0.4 | -0.7736912350 | -0.7736912350 | 0.000000e-00 | 2.169609e-05 |
| 0.6 | -0.5282147935 | -0.5282147940 | 5.000000e-10 | 7.390976e-06 |
| 0.8 | -0.2508144153 | -0.2508144150 | 3.000000e-10 | 7.450581e-07 |

**Table 2:** The result of the proposed method compared Galerkin Method with Septic B-splines (Kasi Viswanadham and Sreenivasulu Ballem, 2015) [5]

| Z | Exact solution | Approximate solution | Absolute Error by the proposed method | GMSB-s Error |
|---|---|---|---|---|
| -0.8 | 1.2912409640 | 1.291240963 | 1.000000e-09 | 4.649162e-06 |
| -0.6 | 0.9034279574 | 0.9034279574 | 0.000000e-00 | 1.329184e-05 |
| -0.4 | 0.5451856792 | 0.5451856793 | 1.000000e-10 | 2.050400e-05 |
| -0.2 | 0.2384031970 | 0.2384031971 | 1.000000e-10 | 9.477139e-06 |
| 0.0 | 0.0000000000 | 1.0000000e-10 | 1.000000e-10 | 2.731677e-06 |
| 0.2 | -0.1589354646 | -0.1589354645 | 1.000000e-10 | 1.458824e-05 |
| 0.4 | -0.2336510054 | -0.2336510052 | 2.000000e-10 | 2.110004e-05 |
| 0.6 | -0.2258569894 | -0.2258569891 | 3.000000e-10 | 1.908839e-05 |
| 0.8 | -0.1434712182 | -0.1434712178 | 4.000000e-10 | 1.342595e-05 |

**Table 3:** The result of the proposed method compared Galerkin Method with Septic B-splines (Kasi Viswanadham and Sreenivasulu Ballem, 2015) [5]

| Z | Exact solution | Approximate solution | Absolute Error by the proposed method | GMSB-s Error |
|---|---|---|---|---|
| 0.1 | 0.9946538262 | 0.9946537928 | 3.340000e-08 | 1.537800e-05 |
| 0.2 | 0.9771222064 | 0.9771221720 | 3.440000e-08 | 4.452467e-05 |
| 0.3 | 0.9449011656 | 0.9449011329 | 3.270000e-08 | 3.331900e-05 |
| 0.4 | 0.8950948188 | 0.8950947907 | 2.810000e-08 | 3.552437e-05 |
| 0.5 | 0.8243606355 | 0.8243606144 | 2.110000e-08 | 9.477139e-06 |
| 0.6 | 0.7288475200 | 0.7288475090 | 1.100000e-08 | 2.586842e-05 |
| 0.7 | 0.6041258121 | 0.6041258152 | 3.100000e-09 | 3.975630e-05 |
| 0.8 | 0.4451081856 | 0.4451082079 | 2.230000e-08 | 3.531575e-05 |
| 0.9 | 0.2459603111 | 0.2459603566 | 4.550000e-08 | 2.214313e-05 |

**Table 4:** The result of the proposed method compared with Reproducing Kernel Hilbert Space Method (Ali *et al.*, 2018) [2]

| Z | Exact solution | Absolute Error by the proposed method | AE (RKHSM) [2] | AE (NPCSM) [3] |
|---|---|---|---|---|
| 0.2 | 0.195424441 | 1.000000e-08 | 3.330000e-08 | 2.433000e-07 |
| 0.4 | 0.358037927 | 1.800000e-08 | 7.031000e-08 | 3.986000e-07 |
| 0.6 | 0.358037927 | 1.700000e-08 | 6.076000e-08 | 4.428000e-07 |
| 0.8 | 0.356086549 | 1.000000e-08 | 2.682000e-08 | 3.328000e-07 |

## Conclusions

In this paper, the reconstructive variational iteration technique using Chebyshev and shifted Chebyshev polynomials of the second kind has been applied successfully to obtain the numerical solutions of tenth order boundary value problems. The reconstruction involves Chebyshev polynomials of the second kind and shifted Chebyshev polynomials of the second kind mixed with variation AL iteration method. The method gives rapidly converging series solutions which occur in physical problem. From tables 1, 2, 3 and 4, it is observed that the proposed method yields a better result when compared with methods in literature. Finally; the numerical results revealed that the present method is also a powerful mathematical tool for the solution of the class of problem considered.

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