International Journal of Mechanical and Thermal Engineering

E-ISSN: 2707-8051
P-ISSN: 2707-8043
IJMTE 2024; 5(1): 25-33
Received: 27-11-2023
Accepted: 30-12-2023

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Mathematical modeling and analysis of steel beam using Reddy beam theory under distributed load

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Abstract
The aim of this study is to analysis of static bending for a steel Reddy beam theory under the influence of a uniformly distributed load. The governing differential equations of steel beam are derived based on a total potential energy principle. Numerical results show us the effect of both wavelength and cross-sectional area on the transverse deflection under simply supported beam. The proposed method validated by comparing the numerical results obtained with those in the literature and found in excellent agreement.

Keywords: Static analysis, steel beam, Reddy beam theory, equilibrium of equations

1. Introduction

Beams and plates are essential components of many engineering structures, as beams and plates are exposed to many different loads that would cause various elastic and plastic deformations. They are used in various fields in the manufacture of aircraft, marine and space ships, and the construction of engineering structures. Santos *et al.* [1] the equations governing the work are derived through the use of isotropic Euler-Bernoulli beam theory stable on an elastic foundation. Numerical study of different boundary conditions is used to determine the transverse deflection. The findings obtained demonstrate a strong concurrence with the existing literature, indicating a distinct impact of the elastic foundation on the transverse deflection [2]. Saba and Mangulkar the desired equations are to be derived based on the hyperbolic shear deformation theory of a cantilever beam which is subjected to a distributed load. The numerical approach is employed to get the maximum deflection and shear stress that match the literature results reported. The governing equations were derived through the utilization of the new shear deformation theory and the proposed ansatz. The transverse deflection and stresses were yielded from numerical means under different boundary conditions and compared with different theories, and this coincided with excellent agreement. In the case of the beam end supports, both natural frequency and transverse diffraction are established. A finite element method was used to get values which are compared with those presented in the literature. SARAÇOĞL *et al.* [5] in their research, started with the governing equations of the Euler-Bernoulli and Timoshenko beams under the influence of a uniformly distributed load. The maximum deflection is determined from numerical analysis and solving the governing equations under different boundary conditions. Ike [6] Governing equations were determined using Timoshenko BEM theory for a thick beam under the influence of a uniformly distributed load along the beam. The transverse deflection of the beam under simply supported boundary conditions is determined using numerical analytic methods and is shown to be in excellent agreement with the values reported in the literature. Author's Name [7] Both the Euler-Bernoulli and Timoshenko beam theories were utilized to develop the governing equations. The numerical Meshless technique was employed to evaluate the transverse deflection and axial stresses, utilizing various boundary conditions. The obtained findings exhibited a high level of concurrence with the existing literature. Ajala *et al.* [8] Different boundary conditions are employed in the finite element approach to determine the transverse deflection and natural frequency. The equation of motion is determined using first order deformation theory. The MATLAB algorithm yielded findings that closely aligned with the existing literature. Shimpi *et al.* [9] the equations controlling the behavior of an isotropic rectangular beam exposed to a uniformly distributed load are determined by the use of shear deformation theory and refined beam theory. Numerical analytical approaches provide insights into the tangential deflection under various boundary conditions and demonstrate a high level of concurrence with existing research. Sayyad [10] The differential equations that describe the behavior of a thick isotropic beam under evenly distributed load are solved in order to determine the transverse deflection, axial bending stress, transverse shear stress, and natural frequencies. The results obtained under simply supported boundary conditions exhibit a high level of concurrence with the findings documented in the existing literature. Nguyen and Nguyen [11] the determination of transverse deflection is achieved by the application of Ritz theory, considering various boundary conditions. The numerical method's verification result demonstrates a high level of agreement. The governing equations are obtained by the application of a Quasi-3D beam theory, considering the effects of a load distributed transversely. Onah *et al*. [12] the equations controlling the behavior of moderately thick and thick beams subjected to a distributed transverse load and axial force are developed using first order shear deformable beam theory. The analytical study determines the buckling load and deflection for different boundaries and this shows a good degree of agreement. Karkon [13] the finite element method is applied, which allows us to obtain the critical buckling load, natural frequency, and transverse deflection by taking into account various boundary conditions being considered. A high degree of agreement between the present findings and previously published research literature is shown by the numbers. The transverse deflection of a beam when the directed transverse load is applied can be analyzed with the help of hyperbolic shear deformation theory governing equations [14]. The findings derived from the numerical analysis of a simply supported beam are juxtaposed with the existing literature and demonstrate a high level of concurrence. KIEN [15] the numerical findings indicate that the inclusion of the nonlinear factor in the local strain expression significantly impacts the precision of the elements in the analysis of large displacement beam and frame constructions. Yesilce and Catal [16] the differential equations governing the motion of a rectangular beam in a state of free vibration were obtained by the use of Bernoulli-Euler, Timoshenko, and Reddy-Bickford beams, as well as the utilization of Hamilton's principle. These equations were derived under various easily supported boundary conditions. The determination of the natural frequency involves the resolution of the governing differential equations, which are subsequently validated by a comparative analysis with other obtained outcomes. Yesilce [17] a comparison is made between the numerical combination methodology and the secant method by solving the governing differential equations, and the single-range and multi-beam natural frequencies derived using Timoshenko beam theory (TBT). The mode forms are visually shown in diagrams. Kacar *et al*. [18] the differential transformation technique is employed to solve the governing differential equations for a simply supported beam. The natural frequency is determined by comparing the numerical results with those reported in the literature, and it is found to be in excellent agreement. Soltani and Mohammadi [19] the numerical outcomes obtained from solving the differential equations governed by Euler-Bernoulli beam theory for a simply supported beam indicate that the increase in the non-local parameter leads to a reduction in the critical buckling stresses, hence stabilizing the beam. Zhu *et al*. [20] the numerical analysis reveals that the critical buckling loads exhibit near solutions and demonstrate a strong agreement with the numerical findings documented in the existing literature. The results indicate that the non-local effect has a significant impact on reducing the critical buckling loads.

2. Model description

Figure 1 shows a schematic view of a steel beam subjected to the distributed transverse load q(x). The steel beam is simply supported and homogeneous, and it is characterized by several parameters: Young's modulus (E), Poisson's ratio (ν), length (L), thickness (h), and width (b). It is assumed that the Poisson's ratio (ν) is constant in the thickness direction.

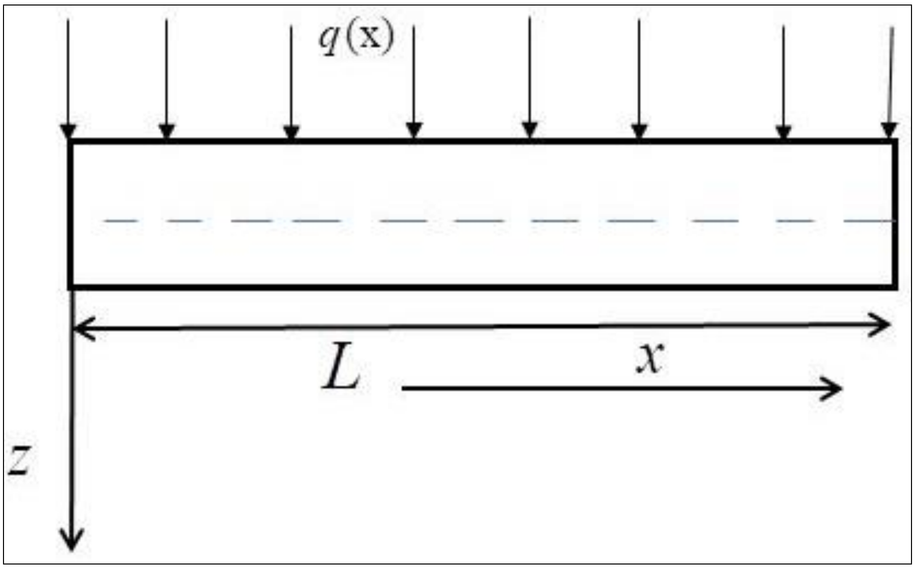


Fig 1: A steel beam subjected to distributed transverse load

We start deriving the required of governing equations of equilibrium for steel beam are derived based on a total potential energy theory. For Reddy beam theory (RBT) the displacements can be written as.

$$u(x,y,z) = z\phi(x) - \alpha z^3\left(\phi(x) + \frac{\partial w(x)}{\partial x}\right) \quad (1a)$$

$$v(x,y,z) = 0 \quad (1b)$$

$$w(x,y,z) = w_0(x) \quad (1c)$$

By using Eqs. (1a) & (1c) the axial and shear strains are given as.

$$\varepsilon_{xx} = z\frac{\partial \phi}{\partial x} - \alpha z^3\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) \quad (2a)$$

$$\gamma_{xz} = \left(\phi + \frac{\partial w_0}{\partial x}\right) - \beta z^2\left(\phi + \frac{\partial w_0}{\partial x}\right) \quad (2b)$$

$$where: \quad \beta = 3\alpha = \frac{4}{h^2}$$

The virtual strain energy (potential energy) of the steel Reddy beam can be written as following.

$$\delta U = \int_0^L \int_A \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz}\right) \mathrm{dA}\,\mathrm{dx} \quad (3)$$

Then,

$$\delta U = \int_0^L \int_A \sigma_{xx}\delta\left(z\frac{\partial \phi}{\partial x} - \alpha z^3\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\right) + \sigma_{xz}\delta\left(\left(\phi + \frac{\partial w}{\partial x}\right) - \beta z^2\left(\phi + \frac{\partial w}{\partial x}\right)\right)\mathrm{dA}\,\mathrm{dx} \quad (4a)$$

$$\delta U = \int_0^L \int_A \sigma_{xx} z\,\delta\frac{\partial \phi}{\partial x} - \alpha\,\sigma_{xx} z^3 \delta\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + \sigma_{xz}\delta\left(\phi + \frac{\partial w_0}{\partial x}\right) - \beta\,\sigma_{xz} z^2 \delta\left(\phi + \frac{\partial w_0}{\partial x}\right)\mathrm{dA}\,\mathrm{dx} \quad (4b)$$

$$\delta U = \int_0^L \int_A \sigma_{xx} z\,\mathrm{dA}\,\delta\frac{\partial \phi}{\partial x} - \alpha\,\sigma_{xx} z^3\,\mathrm{dA}\,\delta\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) +$$

$$\sigma_{xz}\,\mathrm{dA}\,\delta\left(\phi + \frac{\partial w_0}{\partial x}\right) - \beta\,\sigma_{xz} z^2\,\mathrm{dA}\,\delta\left(\phi + \frac{\partial w}{\partial x}\right)\mathrm{dx} \quad (4c)$$

where :

$$M_x = \int z\,\sigma_{xx} dA \qquad , Q_x = \int \sigma_{xz} dA$$

$$P_x = \int z^3 \sigma_{xx} dA \qquad , R_x = \int z^2 \sigma_{xz} dA \quad (5)$$

M_x & Q_x are the bending moment and shear force respectively.

P_x & R_x are the higher order stress resultants can be written respectively. And then.

$$\delta U = \int_0^L M_x \delta \frac{\partial \phi}{\partial x} - \alpha P_x \delta \left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + Q_x \delta \left(\phi + \frac{\partial w}{\partial x} \right) - \beta R_x \delta \left(\phi + \frac{\partial w}{\partial x} \right) \mathrm{dx} \qquad (6a)$$

$$\delta U = \int_0^L M_x \delta \frac{\partial \phi}{\partial x} - \alpha P_x \delta \frac{\partial \phi}{\partial x} - \alpha P_x \delta \frac{\partial^2 w}{\partial x^2} + Q_x \delta \left(\phi + \frac{\partial w_0}{\partial x} \right) - \beta R_x \delta \left(\phi + \frac{\partial w}{\partial x} \right) \mathrm{dx} \qquad (6b)$$

From Eq. (6b) the final virtual strain energy (potential energy) for Reddy beam theory of the static bending.

$$\delta U = \int_0^L \left(M_x - \alpha P_x \right) \delta \frac{\partial \phi}{\partial x} - \alpha P_x \delta \frac{\partial^2 w_0}{\partial x^2} + \left(Q_x - \beta R_x \right) \delta \left(\phi + \frac{\partial w_0}{\partial x} \right) \mathrm{dx} \qquad (7)$$

The external work by applied forces for steel Reddy beam theory (RBT) as follows:

$$\delta W = -\int_0^L q(x)\, \delta w_0 \, dx \qquad (8)$$

The final external work for steel Reddy beam theory (RBT).

Where
q(x): Distributed transverse load
The governing equations of isotropic steel Reddy beam theory (RBT) by using potential energy theory as following:

$$\delta U + \delta W_{ext} = 0 \qquad (9)$$

Substituting Eqs. (7) & (8) into Eq.(9) and setting the coefficients of $\delta\phi \,\&\, \delta w_0$ to zero the final equations of equilibrium of the Reddy beam theory (RBT) are written as following.

$$\delta\phi: \quad \frac{\partial}{\partial x}\left(M_x - \alpha P_x \right) + \left(Q_x - \beta R_x \right) = 0 \qquad (10)$$

$$\delta w: \quad -\alpha \frac{\partial^2}{\partial x^2} P_{xx} + \frac{\partial}{\partial x}\left(Q_x - \beta R_x \right) = q(x) \qquad (11)$$

The Final equations of equilibrium of the Reddy beam theory of the bending analysis.
Where:

$$M_x = \frac{Ebh^3}{12} \frac{\partial \phi}{\partial x} - \frac{Eb\alpha h^5}{80} \left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \qquad (12a)$$

$$P_x = \frac{Ebh^5}{80} \frac{\partial \phi}{\partial x} - \frac{Eb\alpha h^7}{448} \left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \qquad (12b)$$

$$Q_x = G\,bh \left(\phi + \frac{\partial w_0}{\partial x} \right) - \frac{G\,b\,\beta h^3}{12} \left(\phi + \frac{\partial w_0}{\partial x} \right) \qquad (12c)$$

$$R_x = \frac{G\,bh^3}{12} \left(\phi + \frac{\partial w_0}{\partial x} \right) - \frac{Gb\,\beta h^5}{80} \left(\phi + \frac{\partial w_0}{\partial x} \right) \qquad (12d)$$

By substituting Eqs. (12a)-(12b)-(12c)&(12d) into Eqs. (10) & (11) the final the governing of equations for Reddy beam theory (RBT) as following.

$$\frac{Ebh^3}{12}\frac{\partial^2\phi}{\partial x^2}-\frac{Eb\alpha h^5}{40}\frac{\partial^2\phi}{\partial x^2}-\frac{Eb\alpha h^5}{80}\frac{\partial^3 w}{\partial x^3}+\frac{Eb\alpha^2 h^7}{448}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)+$$
$$G\,bh\left(\phi+\frac{\partial w_0}{\partial x}\right)-\frac{G\,b\,\beta h^3}{6}\left(\phi+\frac{\partial w_0}{\partial x}\right)+\frac{G\,b\,\beta^2 h^5}{80}\left(\phi+\frac{\partial w_0}{\partial x}\right)=0 \qquad (13)$$

$$-\frac{E\,\alpha\,h^5}{80}\frac{\partial^3\phi}{\partial x^3}+\frac{E\,\alpha^2\,h^7}{448}\left(\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^4 w}{\partial x^4}\right)+G\,h\left(\frac{\partial\phi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)$$
$$-\frac{G\,\beta h^3}{6}\left(\frac{\partial\phi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+\frac{G\,\beta^2 h^5}{80}\left(\frac{\partial\phi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)=q(x) \qquad (14)$$

3. Solution Methods

The governing equations of the steel Reddy beam theory (RBT) solved of the static bending by using Navier-type solution methods. Isotropic beam subjected to the distributed transverse load q(x). The boundary conditions of the simply-supported Reddy beam theory (RBT) are provided at x=0 and x=L.

The Navier-type solution method for solving the governing equations of simply-supported FGM Reddy beam theory (RBT) $\alpha = n\pi/L$, $w(x)\,\&\,\phi(x)$ are unknowns variables is defined as follows.

$$w(x)=\sum W_m \sin(\alpha x) \qquad (15a)$$
$$\phi(x)=\sum \phi_m \cos(\alpha x) \qquad (15b)$$
$$q(x)=\sum q_m \sin(\alpha x) \qquad (15c)$$

By substituting Eqs. (15a) - (15b) & (15c) into Eqs. (13) & (14) yields the following.

$$\left[-\frac{E\,h^3}{12}\left(\frac{m\pi}{L}\right)^2+\frac{E\,\alpha h^5}{40}\left(\frac{m\pi}{L}\right)^2-\frac{E\,\alpha^2 h^7}{448}\left(\frac{m\pi}{L}\right)^2+G\,h-\frac{G\,\beta h^3}{6}+\frac{G\,\beta^2 h^5}{80}\right]\phi_m-\left[\frac{E\,\alpha h^5}{80}\left(\frac{m\pi}{L}\right)^3-\right.$$
$$\left.\frac{E\,\alpha^2 h^7}{448}\left(\frac{m\pi}{L}\right)^3+G\,h\left(\frac{m\pi}{L}\right)-\frac{G\,\beta h^3}{6}\left(\frac{m\pi}{L}\right)+\frac{G\,\beta^2 h^5}{80}\left(\frac{m\pi}{L}\right)\right]W_m=0 \qquad (16)$$

$$\left[-\frac{E\,\alpha\,h^5}{80}\left(\frac{m\pi}{L}\right)^3+\frac{E\,\alpha^2\,h^7}{448}\left(\frac{m\pi}{L}\right)^3+G\,h\left(\frac{m\pi}{L}\right)-\frac{G\,\beta h^3}{6}\left(\frac{m\pi}{L}\right)+\frac{G\,\beta^2 h^5}{80}\left(\frac{m\pi}{L}\right)\right]\phi_m+\left[\frac{E\,\alpha^2\,h^7}{448}\left(\frac{m\pi}{L}\right)^4\right.$$
$$\left.+G\,h\left(\frac{m\pi}{L}\right)^2-\frac{G\,\beta h^3}{6}\left(\frac{m\pi}{L}\right)^2+\frac{G\,\beta^2 h^5}{80}\left(\frac{m\pi}{L}\right)^2\right]W_m=qm \qquad (17)$$

By using Eqs. (16) & (17), the matrix form of the isotropic Reddy beam theory (RBT) for static analysis is obtained as follows.

$$\begin{bmatrix} -\frac{E\,h^3}{12}\left(\frac{m\pi}{L}\right)^2+\frac{E\,\alpha h^5}{40}\left(\frac{m\pi}{L}\right)^2-\frac{E\,\alpha^2 h^7}{448}\left(\frac{m\pi}{L}\right)^2+G\,h-\frac{G\,\beta h^3}{6}+\frac{G\,\beta^2 h^5}{80} & -\frac{E\,\alpha h^5}{80}\left(\frac{m\pi}{L}\right)^3+\frac{E\,\alpha^2 h^7}{448}\left(\frac{m\pi}{L}\right)^3+G\,h\left(\frac{m\pi}{L}\right)-\frac{G\,\beta h^3}{6}\left(\frac{m\pi}{L}\right)+\frac{G\,\beta^2 h^5}{80}\left(\frac{m\pi}{L}\right) \\ -\frac{E\,\alpha h^5}{80}\left(\frac{m\pi}{L}\right)^3+\frac{E\,\alpha^2 h^7}{448}\left(\frac{m\pi}{L}\right)^3+G\,h\left(\frac{m\pi}{L}\right)-\frac{G\,\beta h^3}{6}\left(\frac{m\pi}{L}\right)+\frac{G\,\beta^2 h^5}{80}\left(\frac{m\pi}{L}\right) & \frac{E\,\alpha^2\,h^7}{448}\left(\frac{m\pi}{L}\right)^4+G\,h\left(\frac{m\pi}{L}\right)^2-\frac{G\,\beta h^3}{6}\left(\frac{m\pi}{L}\right)^2+\frac{G\,\beta^2 h^5}{80}\left(\frac{m\pi}{L}\right)^2 \end{bmatrix}\begin{bmatrix}\phi_m \\ W_m\end{bmatrix}=\begin{bmatrix}0 \\ q_m\end{bmatrix} \qquad (18)$$

3.1 Static analysis

The numerical results determine the static transverse deflections of the simply supported steel Reddy beam theory (RBT) and Navier's-type solution methods. The physical characteristics of the steel beam are as follows Young's modulus (E) =200 Gpa, length (L) = 1m, thickness (h) = 0.1m, and width (b) = 0.1m. The longitudinal wave of the steel beam is m=1.

Maximum transverse deflection of the Reddy beam theory is defined as.

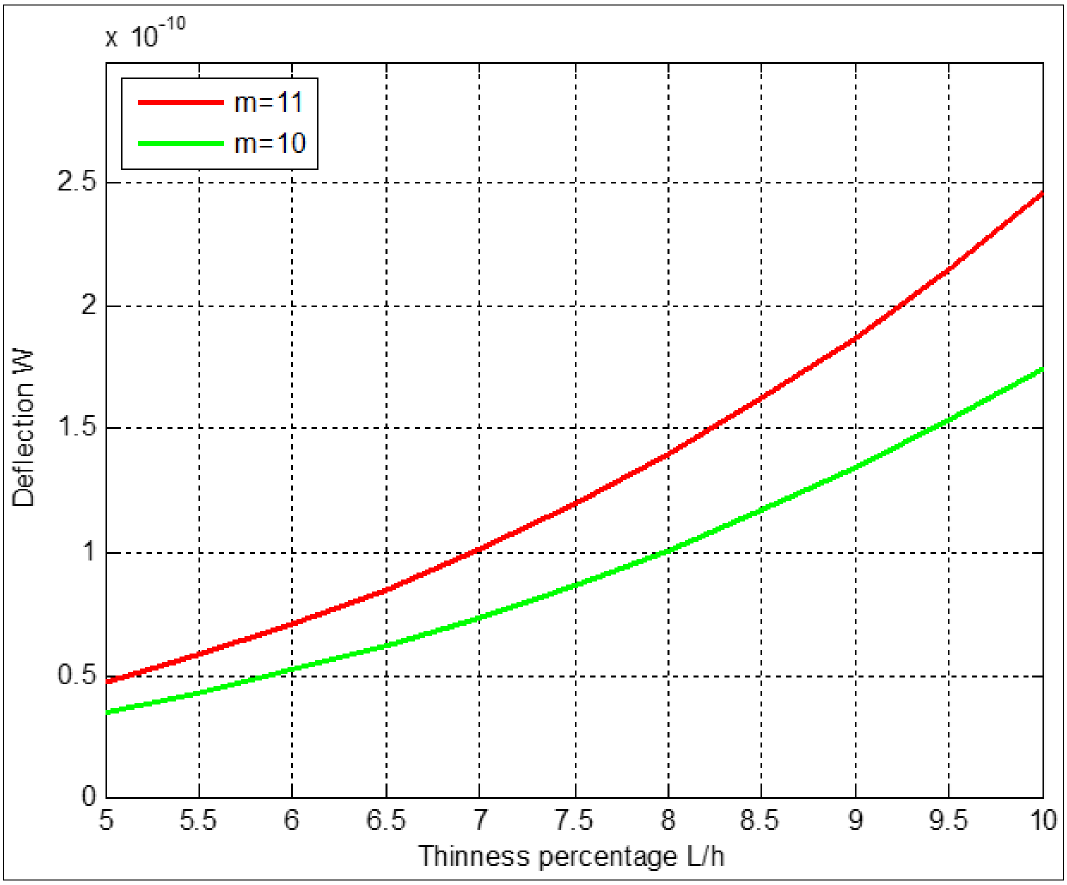
$$\overline{W} = \frac{E * I * w}{q_0 * L^4} \tag{19}$$

Table 1 shows the transverse deflection w with four values of the thickness-to-length ratio (h/L) of simply supported steel Reddy beam theory. Numerical analysis shows the transverse deflections in a excellent agreement with of the solutions of the Shimpi *et al*. [9]. Can be seen from the table 1 that the transverse deflections *w* increases with increasing the values of the thickness-to-length ratio (h/L). This is due to the increase in thickness-to-length ratio (h/L), the cross-sectional area decreases and causes an increase in transverse deflections w.

Table 1: The transverse deflections of steel Reddy beam theory (RBT) with thickness-to-length ratio (h/L)

thickness-to-length ratio (h/L)	Work Present $\overline{W}$ steel RBT	Work Shimpi *et al.* $\overline{W}$ steel RBT
0.01	0.01302	0.01302
0.05	0.01310	0.01310
0.10	0.01335	0.01335
0.15	0.01375	0.01375

Fig 1. It shows the relationship between transverse deflection w and the percentage of thinness for different values of longitudinal wave m of the simply supported Ready Beam theory (RBT). The figure shows that when both the percentage of thinness (h/L) and the longitudinal wave m increase, the transverse deflection also increases



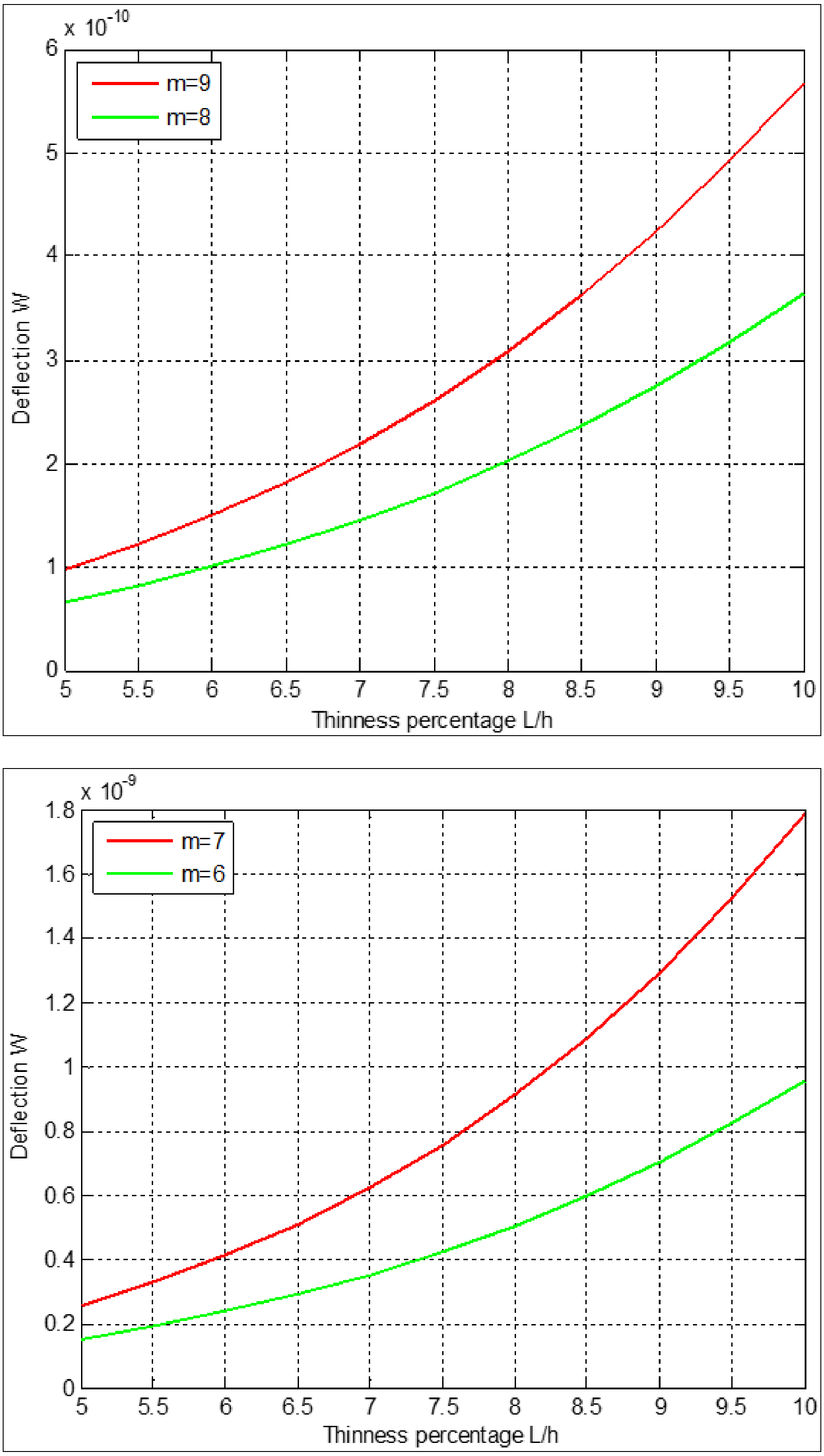


Fig 2: Present effect different values of longitudinal wave m on the transverse deflection w with percentage of thinness (h/L)

Fig 2. The figure shows the variation between the transverse deflection w and both longitudinal wave m and percentage of thinness (h/L) of the simply supported Ready beam theory (RBT). It was found from the figure that as the percentage of thinness (h/L) increases, the transverse deflection increases, the reason for this is that by increasing the percentage of thinness (h/L), the cross-sectional area decreases. It can be seen from the fig 2 that the transverse deflection w increases with increasing longitudinal wave m values.

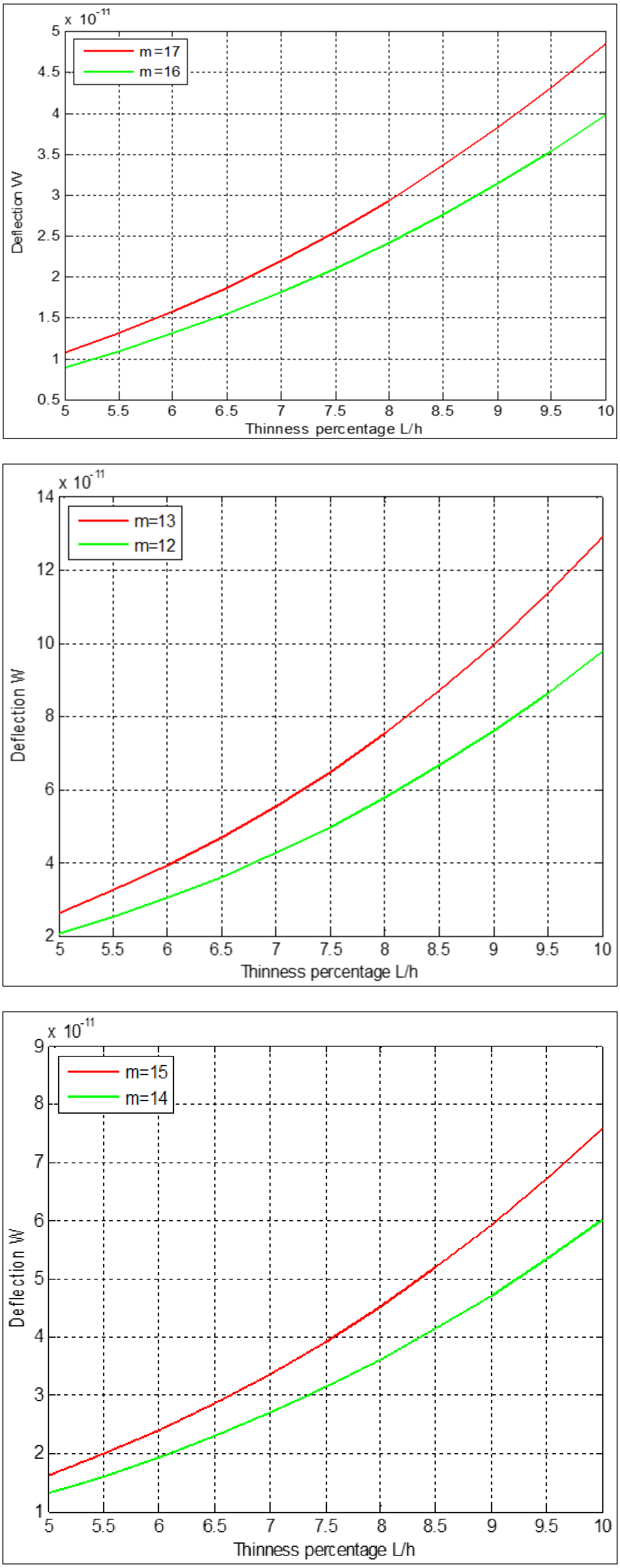


Fig 3: Effect of the longitudinal wave m with percentage of thinness (h/L) on the transverse deflection w of the Reddy beam theory

4. Conclusion

This work aims to analyze the static bending behavior of a steel Reddy beam using theoretical methods. When subjected to a load that is evenly distributed. The equations regulating the equilibrium of a steel beam are developed using the idea of total potential energy. The validity of this approach is established by a comparison between the numerical findings produced and the existing results documented in the literature, revealing a high level of agreement. The findings derived from this study indicate that a rise in both the percentage of thinness and longitudinal wave leads to a corresponding increase in transverse deflection. This phenomenon may be attributed to the reduction in the cross-sectional area of the steel beam as the percentage of thinness increases.

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