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Derivation of linear multistep method for the integration of second order ordinary differential equation via taylor's series approach

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Abstract
In this research, our focus was on deriving an implicit linear multistep method for the direct integration of Initial Value Problems (IVPs) involving second-order ordinary differential equations of the form y'' = f(x, y, y'), employing the Taylor series expansion. Through the analysis and implementation of the Taylor series expansion, we determined the values of y_{n+j} and $x_{n+j}, j = 0(1)2$.. To evaluate the accuracy of the proposed method, we conducted tests using both linear and non-linear second-order ordinary differential equations. The obtained results clearly demonstrated that the proposed method exhibited superior accuracy compared to some existing numerical methods in terms of consistency and convergence. These findings highlight the effectiveness of the proposed method in providing more accurate solutions for second-order ordinary differential equations. The utilization of the Taylor series expansion as an analytical and implementation tool in this research proved to be successful in enhancing the accuracy and reliability of the numerical method. This research contributes to the advancement of numerical techniques for addressing mathematical models in various scientific and engineering domains.

Keywords: Linear multistep method, taylor's series, initial value problems, ordinary differential equations

Introduction

Ordinary differential equations (ODEs) play a fundamental role in modeling various physical phenomena and engineering systems. Solving these equations accurately and efficiently is crucial for understanding and predicting the behavior of dynamic systems. Implicit linear multistep methods (ILMMs) are a class of numerical techniques commonly employed to numerically integrate ODEs. The aim of this paper is to present the derivation of an implicit linear multistep method for the direct integration of second-order ordinary differential equations using Taylor's series approach. By employing Taylor's series expansion, we can approximate the solution of the ODE at a given point based on the values of the solution and its derivatives at nearby points. Implicit methods are particularly useful when dealing with stiff or highly oscillatory systems, where explicit methods may become unstable or require impractically small step sizes. Unlike explicit methods, implicit methods allow for larger step sizes while maintaining stability, making them well-suited for a wide range of ODE problems.

In this research, we focus on the general second order ordinary differential equation of the form:

$$y'' = f(x, y, y'), \;\; y(x_0) = x_0, \;\; y'(x_0) = y'_0 \tag{1}$$

Where prime indicates the derivative with respect to x and f satisfy the Lipschitz Condition of the existence and uniqueness of solution to the equation (1).

Numerous methods and techniques have been developed to address these challenges. For instance, Skwame *et al.* (2017) [19] proposed an order ten implicit one-step hybrid block method for solving stiff second-order ordinary differential equations. In a separate study.

The remarkable contributions made by the aforementioned scientists and researchers have served as a great source of inspiration for our own work. Building upon their achievements, we have been motivated to propose the derivation of an implicit linear multistep method for directly integrating second-order ordinary differential equations using Taylor's series approach. By leveraging this approach, we aim to develop an effective and efficient numerical method that directly addresses the challenges associated with integrating second-order ordinary differential equations.

Specification of the method

The goal of this study is to develop our proposed method using the linear multistep method and the Taylors series approach. The general linear multistep approach takes the following form:

$$y(x) = \sum_{j=0}^{k} \alpha_j y_{n+j} - h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

which can be expressed as:

$$f_{n+j} = f\left(x_{n+j}, y_{n+j}, y'_{n+j}\right) j = 0(1)k-1 \tag{3}$$

Where y_{n+j} is an approximation to $y(x_{n-j})$ and $j = 0(1)k-1$

The coefficient of α and β are constant which do not depend on n subject to the condition$\alpha_k = 1$, $|\alpha_0| + |\beta_0| \neq 0$ are determined to ensure that the methods are consistent and ze equation (2) is defined as:

$$L[y(x):h] = \sum_{j=0}^{k}\left[\alpha_j y(x_n + jh) - h^2 \beta_j y''(x_n + jh)\right] \tag{4}$$

Where $y(x)$ is assumed to have continuous derivatives of sufficiently high order. Expanding in $y(x_n + jh)$ and $y''(x_n + jh)$, where $j = 0,1,2 \cdots k$ in Taylor's series about the point x_n to obtain the following expression:

$$L[y(x):h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + c_3 h^3 y'''(x) \cdots c_n h^n y^n(x) \tag{5}$$

where:

$$\left.\begin{array}{c} c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k \\ c_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + k\alpha_k \\ c_2 = \frac{1}{2!}[\alpha_1 + 2^2\alpha_2 + \cdots + k^2\alpha_k] - [\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k] \\ \vdots \\ c_q = \frac{1}{q!}[\alpha_1 + 2^q\alpha_2 + \cdots\cdots + k^q\alpha_k] - \frac{1}{(q-1)!} \\ [\beta_1 + 2^{q-2}\beta_2 + \cdots\cdots + k^{q-2}\beta_k] \end{array}\right\} \tag{6}$$

Substituting $y = y_{n+j}$ and $f = f_{n+j}$, $j = [0, 1, 2]$ in (2) and compare the obtained result with (6) to yield the following equations:

$$\left.\begin{array}{l} \alpha_0 y_n = \alpha_0 y_n \\ \alpha_1 y_{n+1} = \alpha_1 y_n[x_n + h] \\ = \alpha_1\left[y_n + hy'_n + \left(\frac{h}{2}\right)^2 y''_n + \left(\frac{h}{2}\right)^3 y'''_n + \left(\frac{h}{2}\right)^4 y_n{}^{iv} + \left(\frac{h}{2}\right)^5 y_n{}^{v} \cdots\right] \\ \alpha_2 y_{n+2} = \alpha_2 y_n[x_n + 2h] \\ = \alpha_2\left[y_n + 2hy'_n + \frac{(2h)^2}{2!} y''_n + \frac{(2h)^3}{3!} y'''_n + \frac{(2h)^4}{4!} y_n{}^{iv} + \frac{(2h)^5}{5!} y_n{}^{v} \cdots\right] \\ h^2\beta_0 f_n = h^2\beta_0 y'_n \\ h^2\beta_1 f_{n+1} = h^2\beta_1 y'_n[x_n + h] \\ = h^2\beta_1\left[y'_n + hy''_n + \frac{h^2}{2!} y'''_n + \frac{h^3}{3!} y_n{}^{iv} + \frac{h^4}{4!} y_n{}^{v} + \frac{h^5}{5!} y_n{}^{vi} \cdots\right] \\ h^2\beta_2 f_{n+2} = h^2\beta_2 y'_n[x_n + 2h] \\ = h^2\beta_2\left[y'_n + 2hy''_n + \frac{(2h)^2}{2!} y'''_n + \frac{(2h)^3}{3!} y_n{}^{iv} + \frac{(2h)^4}{4!} y_n{}^{v} + \frac{(2h)^5}{5!} y_n{}^{vi} \cdots\right] \end{array}\right\} \tag{7}$$

We derive the new proposed scheme by solving the aforementioned equations and replacing the values of the unknown parameters in Eq. (2):

$$y_n - 2y_{n+1} + y_{n+2} = \frac{h^2}{216}[18f_n + 180f_{n+1} + 18f_{n+2}] \quad (8)$$

3. Analysis of the proposed method

3.1 Order and Error Constant of the Method

Comparing the Eq. (8) with the expansion of linear operator L which is

$$L[y(x):h] = c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) + c_3 h^3 y'''(x) \cdots c_n h^n y^n(x) \quad (9)$$

According, we say that the method has order P if,

$$c_0 = c_1 = \cdots c_p = c_{p+2} \neq 0 \quad (10)$$

(Henrici 1962) [7] (10) Then c_{p+2} is the error constant and
$c_{p+2} h^{p+2} y^{p+2}(x)$ is the principal local truncation error at the point x_n
Comparing the coefficient of Eq. (9) with Eqn. (6) to obtain thec_i, where $i = 0,1,\dots \dots p+2$ values of the scheme. That is: and $c_6 = c_{p+2} \neq 0$ which implies that the method is of order 4 and error constant $c_{p+2} = -\frac{3}{720}$

3.2 The Consistency of the Method Definition: The method is consistent if it has an order more than or equal to one (Lambert 1973) [16]. Therefore, our scheme is consistent, since it is of order four

3.3 Zero Stability

Definition: A linear multistep method is said to be zero stability if its first characteristic polynomial $p(r)$ satisfies the root condition

3.4 The Convergence of the Method

According to the Dahliquist Theorem (1974) [7] being consistent and zero-stable is a necessary and sufficient condition for a linear multistep approach to be convergent. Because our scheme meets both of the conditions, Eq. (9) is convergent.

4. Implementation of the method

In this section, we apply the suggested technique to solve linear and nonlinear second order ordinary differential equations and compare the numerical results with the existing numerical method.

Problem 1

Consider the following linear initial value problem

$$y'' + y = 0 \; y(0) = 1, \; y'(0) = 1, \; y''(0) = 1 \; h = \frac{1}{10} \quad (11)$$

with analytical solution

$y(x) = cosx + sinx$ (12)

Table 1: Comparism between our proposed method and Ehigie *et al*. (2010) [8]

x	Error in Ehigie *et al.*[4]	Error in New Method
0.1	4.25 E-06	3.8257 E-11
0.2	8.46 E06	4.4526 E-11
0.3	1.26 E-05	2.1504 E-10
0.4	1.66 E-05	2.1286 E-10
0.5	2.05 E-05	2.4179 E-9
0.6	2.41 E-05	2.5875 E-9
0.7	2.75 E-05	2.4750 E-9
0.8	3.07 E-05	3.3468 E-8
0.9	3.35 E-05	3.8737 E-8
1.0	3.35 E-05	3.9527 E-8

Table 1 presents a comparison of the absolute errors resulting from the utilization of Taylor's series expansion and the method proposed by Ehigie *et al.* (2010) [8]. The findings of the study reveal that our technique exhibited superior performance compared to Ehigie *et al.* (2010) [8].

Problem 2

Consider the following nonlinear initial value problem (Kayode & Adeyeye, 2011) [14]

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}, h = \frac{10}{3125} \quad (13)$$

Analytical Solution:

$$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right) \quad (14)$$

Below, Table 2 presents the absolute errors obtained by employing Taylor's series expansion and compares them with the method proposed by Kayode & Adeyeye (2011) [14]. The results indicate that our method outperformed Kayode & Adeyeye (2011) [14] in terms of the achieved outcomes.

Table 2: Comparism between our proposed method and Kayode & Adeyeye (2011) [14]

x	Error in Kayode & Adeyeye (2011) [10]	Error in New Method
0.1	4.831380 E-11	1.334320 E-11
0.2	3.382836 E-09	2.602182 E-10
0.3	1.580320 E-08	3.002608 E-10
0.4	4.333951E-08	6.93432 6E-10
0.5	9.391426E-08	1.549739 E-8

5. Conclusions

This research focused on deriving an implicit linear multistep method for the direct integration of Initial Value Problems (IVPs) involving second-order ordinary differential equations of the form $y'' = f(x, y, y')$, utilizing the Taylor series expansion. By analysing and implementing the Taylor series expansion, the values of y_{n+j} *and* $x_{n+j}, j = 0(1)2$, were determined. The accuracy of the proposed method was assessed by applying it to linear and non-linear second-order ordinary differential equations. The obtained results demonstrated that the proposed method exhibited superior accuracy compared to some existing numerical methods in terms of consistency and convergence. This indicates that the proposed method can provide more reliable and precise solutions for second-order ordinary differential equations. The utilization of the Taylor series expansion as the analytical and implementation tool in this research showcases its effectiveness in enhancing the accuracy and reliability of numerical methods for solving second-order ordinary differential equations. The findings of this research contribute to the advancement of numerical techniques for handling mathematical models in various scientific and engineering domains. Future research can focus on extending the application of the derived method to more complex systems, such as higher-order ordinary differential equations, and investigating its performance under different conditions and problem settings. Further comparative studies with additional existing methods can be conducted to gain a deeper understanding of the advantages and limitations of the proposed method.

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