E-ISSN: 2707-8213
P-ISSN: 2707-8205
IJAE 2022; 3(1): 30-36
Received: 18-10-2021
Accepted: 23-12-2021

Otaide Ikechukwu Jackson
Department of Mathematics, Edwin Clark University Kiagbodo, Nigeria

Ayinde Muhammed Abdullahi
Department of Mathematics, University of Abuja, Abuja, Nigeria

Isiaq Ajimoti Adam
Department of Mathematics, Al-Hikmah University, Ilorin Ilorin, Nigeria

Oyedepo Taiye
Department of Mathematics, Federal College of Dental Technology and Therapy, Enugu, Nigeria

Corresponding Author:
Otaide Ikechukwu Jackson
Department of Mathematics, Edwin Clark University Kiagbodo, Nigeria

Approximate solution of tenth-order boundary value problems using variational iteration techniques via chebyshev-hermite polynomials

Otaide Ikechukwu Jackson, Ayinde Muhammed Abdullahi, Isiaq Ajimoti Adam and Oyedepo Taiye

Abstract
The numerical solution of tenth order boundary value problems was obtained in this paper by employing the Variational iteration technique with Chebyshev-Hermite polynomials. The proposed modification involves constructing Chebyshev-Hermite polynomials for the given boundary value problems, which are then used as basis functions for the approximation. Numerical examples were also provided to demonstrate the proposed method's efficiency and applicability. The calculations were carried out using the Maple 18 software.

Keywords: Approximate solution, boundary value problems, chebyshev-hermite, variational iteration

Introduction

Consider a generalized boundary value problem of the form:

$$a_n \frac{d^j}{dz^j} v + a_{j-1} \frac{d^{j-1}}{dz^{j-1}} v + a_{j-2} \frac{d^{j-2}}{dz^{j-2}} v \ \dots \ a_1 \frac{d}{dz} v + a_0 v = f(z), a < z < b \tag{1}$$

with boundary conditions

$$v(a) = A_1,\ v'(a) = A_2, v''(a) = A_3 \dots v^j(a) = A_i, v(b) = B_1, v'(b) = B_2,$$
$$v''(b) = B_3 \dots v^j(b) = B_r. \tag{2}$$

These types of problems are useful in the mathematical modeling of real-world situations such as heat transfer, thermodynamics, viscoelastic flow, and other engineering sciences. Over the years, several numerical techniques for solving problems of this type have been developed. On a class of second-order boundary value problems, Islam *et al.* [10] used Bernoulli polynomials. Kasi Viswanadham and Sreenivasulu [11] employed the Galerkin method with septic B-splines to solve tenth-order boundary value problems. Ali *et al.* [1] solved tenth-order boundary value problems using the reproducing kernel Hilbert space method. Iqbal [9] used polynomial and non-polynomial cubic spline techniques to solve linear tenth-order boundary value problems. Ghazala [7] presented the homotopy analysis method to solve ninth-order boundary value problems. Mamadu and Njoseh [12] solved the first and second ordinary differential equations using the tau method and the tau-collocation approximation method. Noor and Mohyud-Din [14] used the variational iteration decomposition method to solve eight-order boundary value problems. Zhou and Xu [17] recently proposed a collocation method for numerical solutions of linear and nonlinear singular boundary value problems based on Laguerre wavelets. Ayatullah and Mirna [2] used Hermite polynomials to solve optimal control problems in their Quintic B-spline collocation method for second-order mixed boundary value problems. Biala and Jator [6] presented a family of boundary value methods for second-order boundary value problems.

Su [16] presented a boundary value problem for a coupled system of nonlinear fractional differential equations. For the solution of seventh-order boundary value problems, Shahid and Iftikhar [15] used the variational iteration homotopy perturbation method. He [18] employed the homotopy perturbation method for solving boundary value problems.

Benmezai & Sedkaoui [4] also investigated the existence of a positive solution to the third-order boundary value problem. The work of the aforementioned researchers inspired us to use the variational iteration technique with Chebyshev-Hermite polynomials to solve a tenth-order boundary value problem in this paper. The correction function is corrected for boundary value problems in this proposed method, and the Lagrange multiplier is optimally computed using variational theory. So far, the proposed method has proven to be effective, with encouraging and consistent results. Finally, the solution is presented as an infinite series that typically converges to an accurate solution.

Standard Variational Iteration Technique

Consider the following general differential equation to demonstrate the basic concept of the technique:

$$Lv + Nv - g(z) = 0, \tag{3}$$

Where L is a linear operator, J a nonlinear operator and $g(z)$ is the inhomogeneous term According to variational iteration method, we can construct a correction functional as follows

$$v_{r+1} = v_r(z) + \int_0^z \lambda(t)\left(Lv_r(t) + N\widetilde{v_r(t)} - g(t)\right)dt \tag{4}$$

Where $\lambda(t)$a Lagrange multiplier that can be optimally is identified using a Variational iteration technique. The subscripts represent the nth closest approximation. $\widetilde{v}_r$ is regarded as a restricted variation. i.e., $\widetilde{v}_r$= 0. The relationship (4) is known as a correction functional. Because of the precise identification of the Lagrange multiplier, linear problems can be solved in a single iteration step. In this method, we must optimally determine the Lagrange multiplier $\lambda(t)$optimally, and thus the successive approximation of solution v will be easily obtained by employing the Lagrange multiplier and ourv_0, and the solution is given by

$$\lim_{r-\infty} v_r = v \tag{5}$$

The Lagrange Multiplier, which can be defined as follows,

$$\lambda(t) = (-1)^j \frac{1}{(j-1)!}(t-z)^{j-1} \tag{6}$$

Also plays an important role in determining the solution to the problem.

Chebyshev Polynomials

Chebyshev-Hermite polynomials, also known as "Probabilist's Hermite polynomials," are defined by

$$H_{e_j}(z) = (-1)^j e^{\frac{z^2}{2}} \frac{d^j}{dz^j} e^{-\frac{j^2}{2}} \tag{7}$$

Hence, the first five Chebyshev-Hermite polynomials are as follows:

$$\left.\begin{array}{ll} j = 0: H_{e_0}(z) = & 1 \\ j = 1: H_{e_1}(z) & z \\ j = 2: H_{e_2}(z) = & z^2 - 1 \\ j = 3: H_{e_3}(z) & z^2 - 3z \\ j = 4: H_{e_4}(z) = & z^4 - 6z^2 \\ j = 5: H_{e_5}(z) = & z^5 - 10z^3 + 15 \\ \vdots & \end{array}\right\} \tag{8}$$

Modified Variational Iteration Technique Using Chebyshev And Shifted Chebyshev Polynomials of the Fourth Kind (MVITCP-SCP)

Using (3) and (4), we assume an approximate solution of the form

$$v_{r,J-1}(x) = \sum_{r=0}^{J-1} a_{r,J-1} H_{e_{r,J-1}}(z) \quad (9)$$

Where $H_{e_{r,J-1}}(z)$ are Hermite polynomials, $a_{r,J-1}$ are constants to be determined, and J the degree of approximant. Hence we obtain the following iterative method

$$v_{r+1,J-1}(z) = \sum_{r=0}^{J-1} a_{r,J-1} H_{r,J-1}(z) + \int_0^z \lambda(t) \left(L \sum_{r=0}^{J-1} a_{r,J-1} H_{e_{r,J-1}}(t) + N_l \sum_{r=0}^{J-1} a_{r,J-1} H_{e_{r,J-1}}(t) \right) dt \quad (10)$$

Numerical Applications

In this section, we solved three examples using the proposed method, and the numerical results also demonstrate the proposed scheme's accuracy and efficiency.

Numerical Example 1: Considers the following tenth order boundary value problem [1]

$$v^{(10)} + v = -10(2zsinz - 9cosz), \qquad -1 \le z \le 1 \quad (11)$$

$$v(-1) = v(1) = 0,\ v'(1) = -v'(-1) = 2cos1,\ v''(-1) = v''(1) = 2cos1 - 4sin1 \quad (12)$$

The exact solution for the problem is

$$v = (z^2 - 1)cosz. \quad (13)$$

The correction functional for the boundary value problem (11) and (12) is given as

$$v_{r+1} = v_r(z) + \int_0^z \lambda(t) \left(v^{(10)} + v + 10(2tsint - 9cost) \right) dt$$

Where, $\lambda(t) = \frac{(-1)^{10}(t-z)^9}{9!}$ is the Lagrange multiplier. Applying the modified variational iteration technique using the Chebyshev-Hermite polynomials, we assume an approximate solution of the form

$$v_{j,9}(z) = \sum_{r=0}^{9} a_{r,9} H_{e_{r,9}}(z) \quad (14)$$

Hence, we get the following iterative formula:

$$v_{r+1,J-1}(z) = \sum_{r=0}^{9} a_{r,9} H_{e_{r,9}}(z) + \int_0^z \frac{(t-z)^9}{9!} \left(\frac{d^{10}}{dt^{10}} \left(\sum_{z=0}^{9} a_{r,9} H_{e_{r,9}}(t) \right) + 10(2tsint - 9cost) \right) dt \quad (15)$$

Hence, using (8) iterating and applying the boundary conditions (12) the values of the unknown constants can be determined as follows

$a_{0,9} = -0.627604168$, $a_{1,9} = 0$, $a_{2,9} = -0.406250002$, $a_{3,9} = 0$, $a_{4,9} = -0.192708332$, $a_{5,9} = 0$, $a_{6,9} = 0.0034722223$, $a_{7,9} = 0$, $a_{8,9} = -0.00141369048$, $a_{9,9} = 0$

Consequently, the series solution is given as

$$v(x) = 0.9999999969 + 4.79509654210^{-14}x^{18} - 1.15185403810^{-11}x^{16} + 2.09914644510^{-9}x^{14} - 2.777660867910^{-7}x^{12} + 0.00002507716049x^{10} - 0.00141369048x^8 + 0.04305555574x^6 - 0.5416666673x^4 + 1.499999995x^2 \quad (16)$$

Numerical Example 2: Considers the following tenth order boundary value problem[1]

$$v^{(10)} - (z^2 - 2z)v = 10cosz - (z-1)^3 sinz, \ -1 \le z \le 1 \tag{17}$$

$$v(-1) = 2sin1, \ v(1) = 0, \ v'(-1) = -2cos1 - sin1, \ v'(1) = sin1$$

$$v''(-1) = 2cos1 - 2sin1, v''(1) = 2cos1, v'''(-1) = 2cos1 + 3sin1,$$

$$v'''(1) = -3sin1, v^{(4)}(-1) = -4cos1 + 2sin1, v^{(4)}(1) = -4cos1. \tag{18}$$

The exct solution for the problem is

$$v(z) = (z-1)sinz \tag{19}$$

The correction functional for the boundary value problem (18) and (19) is given as

$$v_{j+1} = v_j(z) + \int_0^z \lambda(t)\left(v^{(10)} - (t^2 - 2t)v - 10cost + (t-1)^3 sint\right)dt \tag{20}$$

where, $\lambda(t) = \frac{(-1)^{10}(t-z)^9}{9!}$ is the Lagrange multiplier.

Applying the modified variational iteration technique using the Chebyshev-Hermite polynomials, we assume an approximate solution of the form

$$v_{j,9}(z) = \sum_{r=0}^{9} a_{r,9} H_{e_{r,9}}(z) \tag{21}$$

Hence, we get the following iterative formula:

$$v_{j+1,J-1}(z) = \sum_{j=0}^{9} a_{r,9} H_{e_{r,9}}(x) + \int_0^z \frac{(t-z)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{r=0}^{9} a_{r,9} H_{e_{r,9}}(t)\right) - (t^2 - 2t)\sum_{i=0}^{9} a_{i,9} H_{e_{i,9}}(t) - 10cost + (t-1)^3 sint\right)dt$$

$$v_{j+1,J-1}(z) = a_{0,9}H_{e_{0,9}}(z) + a_{1,9}H_{e_{1,9}}(z) + a_{2,9}H_{e_{2,9}}(z) + a_{3,9}H_{e_{3,9}}(z) + a_{4,9}H_{e_{4,9}}(z) + a_{5,9}H_{e_{5,9}}(z) + a_{6,9}H_{e_{6,9}}(z) + a_{7,9}H_{e_{7,9}}(z) + a_{8,9}H_{e_{8,9}}(z) + a_{9,9}H_{e_{9,9}}(z) + \tag{22}$$

$$\int_0^z \frac{(t-z)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{r=0}^{9} a_{r,9} H_{e_{r,9}}(t)\right) - (t^2 - 2t)\sum_{r=0}^{9} a_{r,9} H_{e_{r,9}}(t) - 10cost + (t-1)^3 sint\right)dt \tag{23}$$

As a result of (8), iteration, and application of the boundary conditions (18), the values of the unknown constants can be determined as follows

$$a_{0,9} = 0.6041666666, \ a_{1,9} = -0.60677081, a_{2,9} = 0.291666669, a_{3,9} = 0.1006945,$$

$$a_{4,9} = -0.083333333, \ a_{5,9} = -0.00520833, a_{6,9} = 0.0027777779,$$

$$a_{7,9} = 0.00099205, \ a_{8,9} = -0.00019841271, a_{9,9} = -0.0000027557$$

Consequently, the series solution is given as

$$v(z) = 4.306071975\ 10^{-18}z^{21} - 4.1104302\ 10^{-18}z^{20} - 8.2132521\ 10^{-18}z^{19}$$
$$-2.81085018\ 10^{-15}z^{17} - 7.64719518\ 10^{-13}z^{16} + 7.6470244110^{-13}z^{15}$$
$$+1.605905292\ 10^{-10}z^{14} - 1.605903841\ 10^{-10}z^{13} - 2.50521090410^{-8}\ z^{12}$$
$$+2.505210861\ 10^{-8}z^{11} - 0.0000027557z^{9} - 0.0001984127z^{8}$$
$$+0.0001984102x^{7} + 0.00833333378x^{6} - 0.0083332896x^{5}$$
$$-0.1666666706x^{4} + 0.1666665060x^{3} + 1.000000011x^{3}$$
$$-0.9999999215x - \frac{1}{851515702861824000}x^{32} + \frac{1}{362880}x^{10} - 4.4\ 10^{-9} \quad (24)$$

Numerical Example 3: Considers the following tenth order boundary value problem[1]

$$v^{(10)} = -(80 + 19z + z^2)e^z, \quad 0 \le z \le 1 \quad (25)$$

$$v(0) = 0,\ v(1) = 0,\ v''(0) = 0,\ v''(1) = -4e,\ v^{(4)}(0) = -8, v^{(4)}(1) = -16e,$$
$$v^{(6)}(0) = -24,\ v^{(6)}(1) = -36e,\ v^{(8)}(0) = -48,\ v^{(8)}(1) = -64e, \quad (26)$$

The exact solution for the problem is

$$v(z) = z(1 - z)e^z \quad (27)$$

The correction functional for the boundary value problem (25) and (26) is given as

$$v_{j+1} = v_r(z) + \int_0^z \lambda(t)\left(v^{(10)} + (80 + 19t + t^2)e^t\right)dt \quad (28)$$

Where, $\lambda(t) = \frac{(-1)^{10}(t-z)^9}{9!}$ is the Lagrange multiplier

Applying the modified variational iteration technique using the Chebyshev-Hermite polynomials, we assume an approximate solution of the form

$$v_{j,9}(z) = \sum_{r=0}^{9} a_{r,9}H_{e_{r,9}}(z) \quad (29)$$

Hence, we get the following iterative formula:

$$v_{j+1,J-1}(z) = \sum_{r=0}^{9} a_{r,9}H_{e_{r,9}}(z) + \int_0^z \frac{(t-z)^9}{9!}\ \left(\frac{d^{10}}{dt^{10}}\left(\sum_{r=0}^{9} a_{r,9}H_{e_{r,9}}(t)\right) + (80 + 19t + t^2)e^t\right) dt \quad (30)$$

$$v_{j+1,J-1}(z) = a_{0,9}H_{e_{0,9}}(z) + a_{1,9}H_{e_{1,9}}(z) + a_{2,9}H_{e_{2,9}}(z) + a_{3,9}H_{e_{3,9}}(z) + a_{4,9}H_{e_{4,9}}(z)$$
$$+a_{5,9}H_{e_{5,9}}(z) + a_{6,9}H_{e_{6,9}}(z) + a_{7,9}H_{e_{7,9}}(z) + a_{8,9}H_{e_{8,9}}(z) + a_{9,9}H_{e_{9,9}}(z)$$
$$+ \int_0^z \frac{(t-z)^9}{9!}\ \left(\frac{d^{10}}{dt^{10}}\left(\sum_{r=0}^{9} a_{r,9}H_{e_{r,9}}(t)\right) + (80 + 19t + t^2)e^t\right) dt \quad (31)$$

As a result of (8), iteration, and application of the boundary conditions (26), the values of the unknown constants can be determined as follows

$$a_{0,9} = -1.625000000,\ a_{1,9} = -3.268229405, a_{2,9} = -4, a_{3,9} = -2.697917067,$$

$$a_{4,9} = -1.083333333, a_{5,9} = -0.3364584897,\ a_{6,9} = -0.06666666667,$$

$$a_{7,9} = -0.01319446435, a_{8,9} = -0.001190476190,\ a_{9,9} = -0.0001736119192$$

Consequently, the series solution is given as

$$v(z) = \frac{1}{114328101888000} z^{18} - \frac{1}{2032499589120} z^{17} - \frac{1}{93405312000} z^{16} - \frac{1}{6706022400} z^{15}$$

$$- \frac{1}{518918400} z^{14} - \frac{1}{43545600} z^{13} - \frac{1}{3991680} z^{12} - \frac{1}{403200} z^{11} - \frac{1}{45360} z^{10}$$

$$-0.0001736119192z^9 - 0.001190476190z^8 - 0.006944435259z^7 - 0.03333333335z^6 - 0.1250000438z^5 - 0.3333333329z^4 - 0.499999909z^3 - 2.10^{-9} + 0.999999943 + 1.10^{-9} \quad (32)$$

Tables

Table 1: The result of the proposed method compared Galerkin method with septic B-spline [11]

z	Exact solution	Approximate solution	Absolute Error by the proposed method	GMSB-s Error
-0.8	-0.2508144153	-0.2508144152	1.000000e-10	5.483627e-06
-0.6	-0.5282147935	-0.5282147918	1.700000e-09	9.536743e-07
-0.4	-0.7736912350	-0.7736912328	2.200000e-09	8.702278e-06
-0.2	-0.9408639147	-0.9408639122	2.500000e-09	2.980232e-07
0.0	-1.0000000000	-0.9999999969	3.100000e-09	1.955032e-05
0.2	-0.9408639147	-0.9408639122	2.500000e-09	2.920628e-05
0.4	-0.7736912350	-0.7736912328	2.200000e-09	2.169609e-05
0.6	-0.5282147935	-0.5282147918	1.700000e-09	7.390976e-06
0.8	-0.2508144153	-0.2508144152	1.000000e-10	7.450581e-07

Table 2: The result of the proposed method compared with Reproducing Kernel Hilbert Space Method [1]

Z	Exact solution	Approximate solution	Absolute Error by the proposed method	GMSB-s Error
-0.8	1.2912409640	1.291240970	6.000000e-09	4.649162e-06
-0.6	0.9034279574	0.9034279406	1.680000e-08	1.329184e-05
-0.4	0.5451856792	0.5451856549	2.430000e-08	2.050400e-05
-0.2	0.2384031970	0.2384031786	1.840000e-08	9.477139e-06
0.0	0.0000000000	4.4000000e-09	4.400000e-09	2.731677e-06
0.2	-0.1589354646	-0.1589354542	1.040000e-08	1.458824e-05
0.4	-0.2336510054	-0.2336509864	1.900000e-08	2.110004e-05
0.6	-0.2258569894	-0.2258569743	1.510000e-08	1.908839e-05
0.8	-0.1434712182	-0.1434712222	4.000000e-09	1.342595e-05

Table 3: The result of the proposed method compared with Reproducing Kernel Hilbert Space Method [1]

z	Exact solution	Absolute Error by the proposed method	AE (RKHSM) [2]	AE (NPCSM) [3]	AE (PCSM) [3]
0.2	0.195424441	9.800000e-09	3.330000e-08	2.433000e-07	3.982000e-04
0.4	0.358037927	1.680000e-08	7.031000e-08	3.986000e-07	6.663000e-04
0.6	0.358037927	1.720000e-08	6.076000e-08	4.428000e-07	7.598000e-04
0.8	0.356086549	1.160000e-08	2.682000e-08	3.328000e-07	5.885000e-04

Conclusions

The modified Variational iteration technique with Chebyshev-Hermite polynomials was successfully applied in this paper to obtain numerical solutions to tenth order boundary value problems. Chebyshev-Hermite polynomials are combined with Variational iteration techniques in the modification. The method produces rapidly converging series solutions, which are common in physical problems. Tables (1), (2), and (3) show that when compared to methods in the literature, the proposed method produces a better result. Finally, the numerical results demonstrated that the current method is a powerful mathematical tool for solving the class of problems under consideration.

Acknowledgments
The author acknowledges the support and facilities of Edwin Clark University and University of Abuja. The authors thanks anonymous reviewers for their insightful suggestions and comments which led to an improved manuscript

References
1. Ali A, Esra KA, Dumitru B, Mustafa I. New Numerical method for solving tenth order boundary value problems Mathematics. 2018;6(11):1-9.
2. Ayatullah Y, Mirnia MK. Solving optimal control problems by using Hermite polynomials, Computational Methods for Differential Equation. 2020;8(2):314-329.
3. Ayatullah Y. Direct method for solution variationalal problems by using Hermite polynomials optimal control problems by using Hermite polynomials. Boletin da Sociedade Paranaense de Matematica. 2021;39(6):223-237. DOI:10.5269/bspm.39925.
4. Benmezai A, Sedkaoni E. Positive solution for singular third-order boundary value problems on the half line with first-order derivative dependence. Acta Univ. Sapientiae. Math. 2021;13:105-126. DOI:10.3390/ausm-2021-0006.
5. Benbaaziz Z, Djebali S. On a singular multi-point third-order boundary value problems on the half-line. Math. Bohem. 2020;145(3):305-324. DOI:1021136/MB.2019.0084-18.
6. Biala TA, Jator SNA. Family of boundary value methods for systems of second-order boundary value problems. International Journal of Differential Equation; c2017. p. 1-12. DOI:10.1155/2017/2464759.
7. Ghazala A, Maasoomah S. Application of Homotopy analysis method to the solution of ninth order boundary value problems in AFTI-F16 fighters, Journal of the Association of Arab Universities for Basic and Applied Sciences. 2017;24:149-155. DOI:10.1016/j.jaubas.2016.08.002
8. Kikechi CB. On local polynomial regression estimators in finite populations. Int. J. Stats. Appl. Math. 2020;5:58-63.
9. Iqbal MJ, Rehman S, Pervaiz A, Hakeem A. Approximations for linear tenth-order boundary value problems through polynomial and non-polynomial cubic spline techniques. Proc. Pakistan Acad. Sci. 2015;52:389-396.
10. Islam MS, Shirin A. Numerical solutions of a class of second order boundary value problems on using Bernoulli polynomials, Applied Mathematics. 2011;2(9):1059-1067.
11. Kasi Viswanadham KNS, Sreenivasulu B. Numerical solution of tenth order boundary value by problems Galerkin method with septic B-splines, International Journal of Applied Science and Engineering. 2015;13(2):247-260.
12. Mamadu EJ, Njoseh IN. Tau-Collocation Approximation Approach for Solving First and Second Order Ordinary Differential Equations. Journal of Applied Mathematics and Physics. 2016;4:383-390. DOI:10.4236/jamp.2016.42045
13. Mauro D, Rodrigo CV. Chebyshev, Legendre, Hermite and orthonormal polynomials in D Dimensions. Reports on Mathematical Physics. 2018;81(2):243-271. DOI:10.1016/50034-4877(18)30040-5.
14. Noor MA, Mohyud-Din ST. Variational iteration decomposition method for solving eight order boundary value problem, Hindawi Publishing Corporation; c2007. Article ID 19529, DOI:10.1155/2007/19529.
15. Shahid SS, Iftikhar M. Variational Iteration Homotopy Perturbation method for the solution of seventh order boundary value problems; c2013. p. 1-15.
16. Su X. Boundary value problem for a coupled system of nonlinear fractional differential equations, Applied Mathematics Letter. 2009;22:64-69.
17. Zhou F, Xu X. Numerical solutions for the linear and nonlinear singular boundary value problems using Laguerre Wavelets. Advances in Difference Equation. 2016;7:1-15. DOI:10.1186/513662-016-0754.
18. He JH. Homotopy perturbed method for solving bounadry value problems, Physical Letter A. 2006;350(1-2):87-88. DOI:10.1016/physleta.2005.10.005.